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ERRATUM

In Notes and Comments, vol. 44 (1971), p. 241, the comment attributed to Simeon Reich on a paper by Thompson was a badly garbled (by the editor) version of the comment he actually made which was as follows:

In [1] Thompson showed, *inter alia*, that if one knows that the image of the plane by a nonconstant complex polynomial is open, then the Fundamental Theorem of Algebra follows. Perhaps it might be of interest to remark that this fact is an immediate consequence of the closedness of this image and of the connectedness of the plane. This closedness is easily established by the following argument.

Let $p(z)$ be a nonconstant polynomial, and let $w_n \xrightarrow[n\to\infty]{} w$ where $w_n = p(z_n)$. $\{z_n\}$ must be bounded because $z_{n_k} \xrightarrow[k\to\infty]{} \infty$ would imply $p(z_{n_k}) \to \infty$. Thus $\{z_n\}$ has a convergent subsequence $z_{n_j} \xrightarrow[j\to\infty]{} z$. Hence $p(z) = \lim_{j\to\infty} p(z_{n_j}) = \lim_{j\to\infty} w_{n_j} = w$, as required.

Reference

1. R. L. Thompson, Open mappings and the fundamental theorem of algebra, this MAGAZINE, 43 (1970) 39–40.

INEQUALITIES CONCERNING THE AREAS OBTAINED WHEN ONE TRIANGLE IS INSCRIBED IN ANOTHER

J. F. RIGBY, University College, Cardiff, Wales

1. Introduction. Let ABC be a triangle. Let D be a point between B and C, let E be a point between C and A, and let F be a point between A and B. Denote the areas of triangles DEF, AEF, BFD, CDE by p, a, b, c, *and assume without loss of generality that* $a \leqq b \leqq c$. Various authors have proved that $p \geqq a$; references are given in [1], pp. 80, 81. Diananda [2] has proved a stronger result: $p \geqq \sqrt{(ab)}$. Diananda's proof is also given in [1].

We shall prove here an even stronger result: $p \geqq \frac{1}{2}[\sqrt{(a^2 + 8ab)} - a]$ (Theorem 3). Theorem 4 shows that this is the best possible inequality involving only p, a and b.

In Theorems 1 and 2 we shall show that the best possible inequality involving p, a, b and c is

$$\text{(A)} \qquad p^3 + (a + b + c)p^2 - 4abc \geqq 0;$$

the assumption that $a \leqq b \leqq c$ is now unnecessary. We shall also show that equality occurs in (A) if and only if AD, BE and CF are concurrent.

If we write $f(x) = x^3 + (a + b + c)x^2 - 4abc$, then $f'(x) > 0$ if $x \geqq 0$, $f(0) < 0$ and $f(x) \to \infty$ as $x \to \infty$. Hence the equation $f(x) = 0$ has just one positive root $\alpha(a,b,c)$ say, so that (A) can be restated as $p \geqq \alpha(a,b,c)$.

If we denote the area of triangle ABC by d, so that $d = p + a + b + c$, then (A) becomes $p^2 \geqq 4abc/d$.

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If we write $f(x) = x^3 + (a+b+c)x^2 - 4abc$, then $f'(x) > 0$ if $x \geqq 0$, $f(0) < 0$ and $f(x) \to \infty$ as $x \to \infty$. Hence the equation $f(x) = 0$ has just one positive root $\alpha(a,b,c)$ say, so that (A) can be restated as $p \geqq \alpha(a,b,c)$.

If we denote the area of triangle ABC by d, so that $d = p + a + b + c$, then (A) becomes $p^2 \geqq 4abc/d$.

2. A geometrical proof of Theorem 1. In the next section we shall prove (A) by an algebraic method, but the geometrical method by which I first obtained the result shows an interesting application of the affine and projective geometry of conics.

Let DEF be a "fixed" triangle of given area p, and let ABC be a "variable" triangle circumscribed to DEF such that triangles AEF and BFD have given areas a and b respectively (Figure 1). Then A lies on a line Oy parallel to EF, and B lies on a line Ox parallel to FD. As A and B vary on Oy and Ox, C traces out a conic $\mathscr{C}$ passing through E and D. (There are various notations for the proof of this result, such as: $EC \wedge A \overset{F}{\wedge} B \wedge DC$.) By taking various special positions for A, we see that $\mathscr{C}$ passes through O, G and H also, where $G = Oy \cap FD$ and $H = Ox \cap EF$.

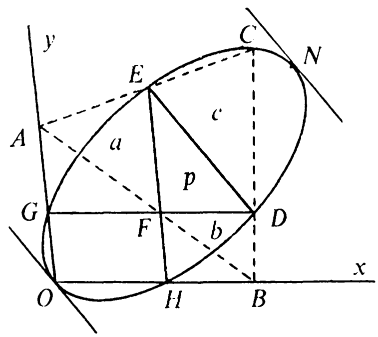

FIG. 1.

If $\mathscr{C}$ is an ellipse, as shown in Figure 1, then the maximum value of c (the area of triangle CDE) occurs when C is at the point N where the tangent to $\mathscr{C}$ is parallel to DE.

Consider the hexagon $OOGDEH$ inscribed in $\mathscr{C}$; OG is parallel to EH, and GD is parallel to HO, hence by Pascal's theorem DE is parallel to OO, the tangent at O. Hence O and N are diametrically opposite points of $\mathscr{C}$.

Now OG and HE are parallel chords of $\mathscr{C}$, so the join of their midpoints passes through the center of $\mathscr{C}$. Similarly the join of the midpoints of OH and GD passes through the center.

Let us take Ox and Oy as coordinate axes, and suppose without loss of generality that F, D, E have coordinates $(1,1)$, $(1+d,1)$ $(1,1+e)$. We can find the center of $\mathscr{C}$ by the method described in the previous paragraph, and then N is found to have coordinates $(2(d+2)/(4-de), 2(e+2)/(4-de))$. Assume for the moment that $4-de>0$, so that $\mathscr{C}$ is an ellipse. Using the determinantal formula for the area of a triangle, and denoting the area of triangle NDE by n, we find that $p=\lambda de$, $b=\lambda d$, $a=\lambda e$, $n=\lambda de(de+d+e)/(4-de)$, where λ is a constant of proportion. Hence $c \leqq \lambda de(de+d+e)/(4-de)$, so $c(4-de) \leqq \lambda de(de+d+e)$. This inequality is still true if $4-de \leqq 0$. We can rewrite the inequality as

$$c\left(\frac{4ab}{p}-p\right) \leqq p(p+b+a),$$

i.e.,

$$p^3 + (a + b + c)p^2 - 4abc \geqq 0.$$

3. Algebraic proofs.

THEOREM 1. *Using the same notation as in the introduction,*

$$p^3 + (a + b + c)p^2 - 4abc \geqq 0,$$

with equality if and only if AD, BE and CF are concurrent.

PROOF. Let BC, CA and AB be divided at D, E and F respectively in the ratios $x:x'$, $y:y'$ and $z:z'$, where $x + x' = y + y' = z + z' = 1$, and let ABC have area d. Then $a = y'zd$, $b = z'xd$, $c = x'yd$, and $p = (1 - y'z - z'x - x'y)d = [xyz + (1 - z + yz - x + zx - y + xy - xyz)]d = (xyz + x'y'z')d$. Hence

$$p^3 + (a + b + c)p^2 - 4abc = p^2d - 4abc = (xyz - x'y'z')^2d^3 \geqq 0.$$

Equality occurs if and only if

$$\frac{x}{x'} \cdot \frac{y}{y'} \cdot \frac{z}{z'} = 1,$$

i.e., if AD, BE and CF are concurrent, by Ceva's theorem

THEOREM 2. *If p, a, b and c are any positive real numbers satisfying the inequality* (A), *and if ABC is any triangle with area* $p + a + b + c$, *then there exist just k different positions for an inscribed triangle DEF such that the triangles DEF, AEF, BFD and CDE have areas p, a, b, and c, where* $k = 1$ *or* 2 *according as we have equality or strict inequality in* (A).

Proof. Since only the ratios of p, a, b and c are important, we may suppose without loss of generality that $p + a + b + c = 1$, so that p, a, b and c all lie between 0 and 1. Let s denote the positive square root of $p^3 + (a + b + c)p^2 - 4abc$.

We must solve the equations $a = (1-y)z$, $b = (1-z)x$, $c = (1-x)y$, using the notation of the proof of Theorem 1. Elimination of y and z gives $(1 - a)x^2 + (a - b + c - 1)x + b(1 - c) = 0$, from which we obtain $x = (p + 2b \pm s)/2(1 - a)$. Then $y = (p + 2c \pm s)/2(1 - b)$ and $z = (p + 2a \pm s)/2(1 - c)$, where the sign before s in y and z is the same as the sign in x. It is easily verified that x, y and z all lie between 0 and 1, whichever sign occurs before s. Hence we obtain two distinct inscribed triangles DEF satisfying the required conditions, unless $s = 0$ when there is only one such triangle.

I originally used a geometrical method to obtain Theorem 3, but the result is more easily deduced from Theorem 1.

THEOREM 3. *Using the same notation as in the introduction, if* $a \leqq b \leqq c$ *then* $p \geqq \frac{1}{2}[\sqrt{(a^2 + 8ab)} - a]$.

Proof. By Theorem 1 we have

$$p^3 + (a + b + c)p^2 - 4abc \geqq 0,$$

hence

$$c(4ab - p^2) \leqq p^3 + (a + b)p^2.$$

Hence, if $4ab - p^2 \geqq 0$,

$$b(4ab - p^2) \leqq p^3 + (a + b)p^2.$$

This last inequality is trivially true if $4ab - p^2 < 0$, and hence in all cases

$$p^3 + (a + 2b)p^2 - 4ab^2 \geqq 0,$$

i.e.,

$$(p + 2b)(p^2 + ap - 2ab) \geqq 0,$$

so that

$$p^2 + ap - 2ab \geqq 0.$$

Hence

$$(2p + a)^2 \geqq (a^2 + 8ab),$$

from which we easily deduce that

$$p \geqq \tfrac{1}{2}[\sqrt{(a^2 + 8ab)} - a].$$

THEOREM 4. *Let p, a and b be any positive real numbers such that* $b \geqq a$ *and* $p \geqq \frac{1}{2}[\sqrt{(a^2 + 8ab)} - a]$. *If* (i) $p^2 \geqq 4ab$ *and c is any number such that* $a \leqq b \leqq c$, *or if* (ii) $p^2 < 4ab$ *and c is any number such that* $a \leqq b \leqq c \leqq [p^3 + p^2(a+b)]/[4ab - p^2]$, *then there exists a triangle ABC and an inscribed triangle DEF such that the triangles DEF, AEF, BFD and CDE have areas p, a, b and c (where* $a \leqq b \leqq c$*).*

Proof. We deduce from the second of the given inequalities, by performing the steps of the proof of Theorem 3 in reverse order, that

$$b(4ab - p^2) \leqq p^3 + (a + b)p^2.$$

Hence in case (ii) we have $b \leqq [p^3 + (a + b)p^2]/[4ab - p^2]$, so that values of c certainly exist satisfying the inequalities given under (ii); for such values of c we have

$$c(4ab - p^2) \leqq p^3 + (a + b)p^2.$$

This last inequality is trivially true in case (i); since the inequality is equivalent to (A) we can apply Theorem 2 in both cases to obtain the result.

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1. O. Bottema, R. Ž. Djordjević, R. R. Janić, D. S. Mitrinović, P. M. Vasić, Geometric Inequalities, Wolters-Noordhoff, Groningen, 1968.
2. P. H. Diananda, Solution to Problem 4908, Amer. Math. Monthly, 68 (1961) 368.

THE PARTIAL FRACTION DECOMPOSITION OF A RATIONAL FUNCTION

HUGH J. HAMILTON, Pomona College

The student knows how to show that

$$\frac{8x^3+16x^2+12x+8}{x^4+2x^3+2x^2} = \frac{2}{x} + \frac{4}{x^2} + \frac{6x+8}{x^2+2x+2} = \frac{2}{x} + \frac{4}{x^2} + \frac{3+i}{x+1+i} + \frac{3-i}{x+1-i},$$

the first decomposition being in the real field and the second in the complex. The following easy way to show that such decompositions of rational functions always exist and are unique seems not to be as readily available as it ought to be. The basic theorem in the complex field is given as Theorem 1.

THEOREM 1. *If $P(x)$ and $R(x)$ are polynomials, a is a complex number for which $R(a) \neq 0$, and n is a positive integer, then there are a unique constant A and a unique polynomial $S(x)$ for which*

$$\frac{P(x)}{(x-a)^n R(x)} = \frac{A}{(x-a)^n} + \frac{S(x)}{(x-a)^{n-1}R(x)}. \tag{1}$$

Moreover, $A \neq 0$ if $(x-a)$ is not a factor of $P(x)$.

(Equality is of course to be understood as applying for precisely those values of x for which all expressions involved have meaning, and the function that vanishes identically is here classed as a polynomial, though without degree.)

Proof. Using the factor theorem for polynomials, we need only observe that

$$\frac{P(x)}{(x-a)^n R(x)} - \frac{A}{(x-a)^n} = \frac{P(x)-AR(x)}{(x-a)^n R(x)} = \frac{(x-a)S(x)}{(x-a)^n R(x)}$$

for some polynomial $S(x)$ if A can be so chosen that the polynomial $P(x) - AR(x)$ vanishes when $x = a$ – which indeed it can, namely, as $P(a)/R(a)$. Since this is the only possible choice of A, the uniqueness of each of A and $S(x)$ follows. Finally, if $A = 0$, solving (1) for $P(x)$ shows that $(x-a)$ is a factor of $P(x)$ – a result that also follows from the expression just obtained for A and (again) the factor theorem.

Breaking down the last term of (1) similarly, we reduce the right hand side of (1) to

$$\frac{A}{(x-a)^n} + \frac{B}{(x-a)^{n-1}} + \frac{T(x)}{(x-a)^{n-2}R(x)},$$

where B is a certain unique constant and $T(x)$ is a certain unique polynomial. After a total of n such operations, we obtain

$$\frac{P(x)}{(x-a)^n R(x)} = \frac{A}{(x-a)^n} + \frac{B}{(x-a)^{n-1}} + \cdots + \frac{C}{x-a} + \frac{U(x)}{R(x)}, \tag{2}$$

where $A, B, \cdots, C$ are certain unique constants, with $A \neq 0$ if $(x-a)$ is not a factor of $P(x)$, and $U(x)$ is a certain unique polynomial. Next, if there is a complex number b for which $R(x) = (x-b)^m V(x)$, where $V(x)$ is a polynomial with $V(b) \neq 0$ and m is a positive integer, we break down the last term in (2) in similar fashion. Continuing in this manner, we arrive at a familiar result, expressed as Theorem 2.

THEOREM 2. *If $P(x)$ and $Q(x)$ are polynomials, then*

$$
\begin{aligned}
\frac{P(x)}{Q(x)} &= \frac{a_{11}}{x-x_1} + \frac{a_{12}}{(x-x_1)^2} + \cdots + \frac{a_{1n_1}}{(x-x_1)^{n_1}} \\
&+ \frac{a_{21}}{x-x_2} + \frac{a_{22}}{(x-x_2)^2} + \cdots + \frac{a_{2n_2}}{(x-x_2)^{n_2}} \\
&+ \cdots \\
&+ \frac{a_{m1}}{x-x_m} + \frac{a_{m2}}{(x-x_m)^2} + \cdots + \frac{a_{mn_m}}{(x-x_m)^{n_m}} \\
&+ W(x),
\end{aligned}
\tag{3}
$$

where the x_j are the (distinct) zeros of $Q(x)$, the n_j are the respective orders of these zeros, the a_{jk} are certain unique constants, with none of the a_{jn_j} zero if $P(x)$ and $Q(x)$ are relatively prime, and $W(x)$ is a certain unique polynomial.

Observing that all terms in (3) except possibly $P(x)/Q(x)$ and $W(x)$ tend to zero as x becomes numerically infinite, and recalling the behavior of polynomials for numerically large values of the argument, we see that $W(x) = 0$ if the degree p of $P(x)$ is less than the degree q of $Q(x)$ and that the degree of $W(x)$ is $p-q$ if $p \geqq q$.

Let us now suppose that the polynomials $P(x)$ and $Q(x)$ of Theorem 2 are real. Then $Q(x)$ is the product of (i) a constant, (ii) real linear factors of the form $x-a$, and (iii) real quadratic factors of the form x^2+bx+c that are not factorable in the real field. The linear factors lead, as in the proof of Theorem 2, to terms like those on the right hand side of (3), where the a_{jk} are of course now real. To obtain the terms in the decomposition exhibited in elementary calculus that correspond to the nonfactorable quadratics, we find useful a result analogous to Theorem 1; this is presented as Theorem 3. Repeated applications of this result lead to the familiar terms in the decomposition.

In proving Theorem 3, we use bars to denote conjugacy and make use of the facts that (a) the linear factors of x^2+bx+c are of the forms $x-\alpha$ and $x-\bar{\alpha}$, where α is not real; (b) the difference between a complex number and its conjugate is pure imaginary; (c) $\overline{T(\alpha)} = T(\bar{\alpha})$ if $T(x)$ is a real rational function of x; (d) the conjugate of a product is the product of the conjugates; (e) $\overline{(\bar{\alpha})} = \alpha$; and (f) the ratio between two pure imaginaries is real.

THEOREM 3. *Let the zeros of the nonfactorable quadratic x^2+bx+c be α and $\bar{\alpha}$. If $P(x)$ and $R(x)$ are real polynomials with $R(\alpha) \neq 0$ (whence also $R(\bar{\alpha}) \neq 0$)*

and n is a positive integer, then there are unique real constants A and B and a unique real polynomial $S(x)$ for which

$$(4)\qquad \frac{P(x)}{(x^2+bx+c)^n R(x)} = \frac{Ax+B}{(x^2+bx+c)^n} + \frac{S(x)}{(x^2+bx+c)^{n-1}R(x)}.$$

Moreover, not both A and B are zero if $(x-\alpha)$ is not a factor of $P(x)$ (whence also $(x-\bar{\alpha})$ is not such a factor).

Proof.

$$\frac{P(x)}{(x^2+bx+c)^n R(x)} - \frac{Ax+B}{(x^2+bx+c)^n} = \frac{P(x)-(Ax+B)R(x)}{(x^2+bx+c)^n R(x)}$$

$$= \frac{(x^2+bx+c)S(x)}{(x^2+bx+c)^n R(x)}$$

for some polynomial $S(x)$ if A and B can be so chosen that the polynomial $P(x)-(Ax+B)R(x)$ vanishes when $x=\alpha$ and when $x=\bar{\alpha}$. But they *can* be so chosen, for the following reasons. The conditions are that $P(\alpha)-(A\alpha+B)R(\alpha)=0$ and $P(\bar{\alpha})-(A\bar{\alpha}+B)R(\bar{\alpha})=0$, or – denoting the real rational function $P(x)/R(x)$ by $T(x)$ – that

$$(5)\qquad \begin{aligned} A\alpha + B &= T(\alpha) \text{ and}\\ A\bar{\alpha}+B &= T(\bar{\alpha}), \end{aligned}$$

and the determinant of the coefficients of A and B in (5) is $\alpha-\bar{\alpha}\neq 0$. Since moreover the solutions of (5) are $A=[T(\alpha)-T(\bar{\alpha})]/(\alpha-\bar{\alpha}) = [T(\alpha)-\overline{T(\alpha)}]/(\alpha-\bar{\alpha})$ and $B=[\alpha T(\bar{\alpha})-\bar{\alpha}T(\alpha)]/(\alpha-\bar{\alpha}) = [\alpha T(\bar{\alpha})-\overline{\alpha T(\bar{\alpha})}]/(\alpha-\bar{\alpha})$, it follows that A and B are real. Since the values of A and B just obtained are the only possible ones, the uniqueness of each of A, B, and $S(x)$ follows. Finally, if $A=B=0$, solving (4) for $P(x)$ shows that $(x-\alpha)$ is a factor of $P(x)$.

PROBLEM. Is the decomposition displayed in (4) valid in case the quadratic x^2+bx+c *is* factorable in the real field? (The case in which this quadratic is a perfect square may require special attention.)

ANOTHER PROOF OF TEPPER'S IDENTITY

F. J. PAPP, University of Lethbridge

1. Introduction. In this note we give an elementary proof of an identity for $r!$.

(*) *If r is any positive integer and x any real number then*

$$r! = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (x-i)^r.$$

and n is a positive integer, then there are unique real constants A and B and a unique real polynomial S(x) for which

$$\frac{P(x)}{(x^2+bx+c)^n R(x)} = \frac{Ax+B}{(x^2+bx+c)^n} + \frac{S(x)}{(x^2+bx+c)^{n-1}R(x)}. \tag{4}$$

Moreover, not both A and B are zero if $(x-\alpha)$ is not a factor of $P(x)$ (whence also $(x-\bar{\alpha})$ is not such a factor).

Proof.

$$\frac{P(x)}{(x^2+bx+c)^n R(x)} - \frac{Ax+B}{(x^2+bx+c)^n} = \frac{P(x)-(Ax+B)R(x)}{(x^2+bx+c)^n R(x)}$$

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for some polynomial $S(x)$ if A and B can be so chosen that the polynomial $P(x)-(Ax+B)R(x)$ vanishes when $x=\alpha$ and when $x=\bar{\alpha}$. But they *can* be so chosen, for the following reasons. The conditions are that $P(\alpha)-(A\alpha+B)R(\alpha)=0$ and $P(\bar{\alpha})-(A\bar{\alpha}+B)R(\bar{\alpha})=0$, or – denoting the real rational function $P(x)/R(x)$ by $T(x)$ – that

$$\begin{aligned} A\alpha + B &= T(\alpha) \text{ and} \\ A\bar{\alpha} + B &= T(\bar{\alpha}), \end{aligned} \tag{5}$$

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M. Tepper conjectured this result from a consideration of numerical data which he gave in [3] and Calvin T. Long gave a proof of it in [2]. The result is also implicit in problem 20 of Feller [1, p. 285].

The proof which follows is completely elementary in the sense that only the familiar properties of binomial coefficients and two elementary theorems from analysis are required.

2. Preliminaries. As usual, if $0 \leqq i \leqq n$, $\binom{n}{i}$ will denote the number $n!/i!(n-i)!$ (with the convention that $0! = 1$).

CONVENTION. If $i > n$ or $i < 0$ we define $\binom{n}{i}$ to be zero.

This convention allows unrestricted use of the familiar identity $\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$. That is, we are freed from the necessity of keeping watch over the range of i.

Notation. For k a positive integer, let

$$F_k(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (x-i)^k.$$

3. Proofs.

THEOREM 1. *For each positive integer n, $F_n(x)$ is a constant function.*

Proof. $F_1(x) = x - (x-1) = 1$.

Suppose that $F_k(x)$ is a constant function for all positive integers $k < n$. Thus for each k and all x there is a number C_k such that $F_k(x) = C_k$.

If x is not an integer, compute the derivative of F_n at x.

$$\begin{aligned} F_n'(x) &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} n(x-i)^{n-1} = n \sum_{i=0}^{n} (-1)^i \left[\binom{n-1}{i} + \binom{n-1}{i-1}\right](x-i)^{n-1} \\ &= n\left[\sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i}(x-i)^{n-1} + \sum_{j=0}^{n-1} (-1)^{j+1}\binom{n-1}{j}(x-1-j)^{n-1}\right] \\ &= n[F_{n-1}(x) - F_{n-1}(x-1)] = 0. \end{aligned}$$

Thus, $F_n(x)$ is constant on each interval of the form $(m, m+1)$ where m is an integer. Moreover, since $F_n(x)$ is clearly continuous for all x, there is a number C_n for which $F_n(x) = C_n$ for all x (including integer x).

LEMMA. *If n is a given positive integer and i is any integer then*

$$(n-i)\binom{n-1}{i-1} - i\binom{n-1}{i} = 0.$$

Proof. The result follows directly from the definition of $\binom{n}{k}$ if $1 \leqq i \leqq n$ and from the convention otherwise.

THEOREM 2. *If n is a positive integer, $C_n = n!$.*

Proof. $C_1 = 1 = 1!$

We show that $C_n = n \cdot C_{n-1}$.

$$F_n(n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^n$$

$$= \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^n$$

$$= n \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} (n-i)^{n-1} - \sum_{i=0}^{n-1} i(-1)^i \binom{n-1}{i} (n-i)^{n-1} + \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i-1} (n-i)^n$$

$$= nF_{n-1}(n) + \sum_{i=0}^{n-1} (-1)^i (n-i)^{n-1} \left[(n-i) \binom{n-1}{i-1} - i \binom{n-1}{i} \right]$$

$$= nC_{n-1}.$$

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1. W. Feller, An Introduction to Probability Theory and its Applications, vol. I, 3rd ed., Wiley, New York, 1968.
2. C. T. Long, Proof of Tepper's factorial conjecture, this MAGAZINE, 38 (1965) 304–305.
3. M. Tepper, A factorial conjecture, this MAGAZINE, 38 (1965) 303–304.

KAPREKAR'S ROUTINE WITH FIVE-DIGIT INTEGERS

CHARLES W. TRIGG, San Diego, California

Kaprekar's routine consists of rearranging the digits (not all alike) of an integer, N_0, to form the largest and smallest possible integers, finding their difference, N_1, and applying this ordering-subtraction operation (OSO) to N_1 and to the subsequent differences until a self-producing integer or a regenerative loop is obtained. In a system with base r, if all integers having n digits (not all alike) lead to a single one of these terminal situations, then that terminal situation is said to be unanimous.

When this routine is applied to any four-digit (not all alike) integer in the decimal system, the self-producing 6174 is eventually reached [**1**, **2**, **3**]. The routine has been extended to other systems of notation [**4**, **5**], and predictive indices have been developed in some of these systems [**6**]. The *predictive index* for any chosen starting integer establishes the number, m, of OSO's necessary to proceed from that integer to a terminal situation. The investigation has been extended to two-digit [**7**] and three-digit [**8**] integers in various bases.

Herein, the investigation is extended to five-digit integers for bases $r < 13$. Two-digit predictive indices are developed in those systems of notation.

The examination procedure. Since the first step of an OSO is to arrange the

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$$= nF_{n-1}(n) + \sum_{i=0}^{n-1} (-1)^i (n-i)^{n-1} \left[(n-i) \binom{n-1}{i-1} - i \binom{n-1}{i} \right]$$

$$= nC_{n-1}.$$

References

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The examination procedure. Since the first step of an OSO is to arrange the

Base Three		Base Five			
22221		44441		44443	
12222		14444		34444	
02222		24442		04444	
22220		44422	44421	44440	44442
02222		22444	12444	04444	24444
12221		21423	31422	34441	14443
22211	22200	44431	44432		
11222	00222	13444	23444		
10212	21201	30432	20433		
22110		44430		44433	44400
01122		03444		33444	00444
20211		40431		10434	43401
		44410	44420		
		01444	02444		
		42411	41421		
		44411			
		11444			
		32412			

TABLE I — Operational Flow Charts

digits of N_0 in descending order of magnitude, only ordered integers $abcde$ with $e < a$ need be examined.

The five digit integers $abcde$ and $\overline{a+p}\ \overline{b+q}\ c\ \overline{d+q}\ \overline{e+p}$ have the same N_1. Furthermore, N_1 is independent of c. Hence it is necessary to examine only the integers of the form $\overline{r-1}\ \overline{r-1}\ \overline{r-1}\ d\ e$, where $e < r-1$. These $r(r+1)/2-1$ or $(r-1)(r+2)/2$ integers represent the entire field of five-digit (not all alike) integers. Each one may be considered to be the representative (rep) of all five-digit integers that have the same N_1 that it does. Any ordered five-digit integer can be converted into its rep by addition of such multiples of 10001, of 1010 and of 100 as will change each of the three digits on its left to $r-1$. For example, in the decimal system, the rep of 64210 is $64210 + 30003 + 5050 + 700$ or 99963.

To determine the effect of repeated OSO's upon every five-digit integer in a system of notation with base r, it is necessary only to compute the N_1's of the $(r-1)(r+2)/2$ reps. The computations which yield N_1's that have a common rep are grouped together. They are arranged with the N_1's placed under a common subtraction line. Their common rep is put under the N_1 on the left. Thus the $(r-1)(r+2)/2$ computations fall into a flow chart such as those for bases three and five in Table I.

In base three, the self-producing 20211 is unanimous.

In base five, the four-element regenerative loop, 30432, 40431, 42411, 32412, $\cdots$, is unanimous. The application of an OSO to any member of a regenerative loop produces the next member.

In other bases, self-producing integers, regenerative loops, or combinations thereof are the terminal situations.

m	Δ		f
4	10		1
3	20		1
2	11	22	2
1	21		1
			5

TABLE II — Base Three

m	Δ								f
3	10	30							2
2	11	20	21	22	32	40	42	44	8
1	31	33	41	43					4
									14

TABLE III — Base Five

The predictive index. All ordered integers, *abcde*, which have the same rep, have the same values of $a-e$ and $b-d$. Thus the two-digit integer $\overline{a-e}\,\overline{b-d}$ may be used as a *predictive index*, Δ, of the number, m, of applications of the OSO needed to move from a chosen starting integer to a self-producing integer or to a member of a regenerative loop. The Δ's of the various reps in Table I have been tabulated in Tables II and III against the corresponding m's.

In the tables, f is the number of indices whose corresponding numbers require the indicated m OSO's to proceed to a terminal situation. The total of the f's is the same as the number of reps in the system.

Indeed, the predictive index of any five-digit integer, ordered or unordered, actually is the difference of its largest and smallest digits followed by the difference of the largest and smallest of the three remaining digits. Thus, the m of an integer may be located in the table without first converting the integer into its rep. For example, the Δ of 21011 is 20. Reference to Table II shows that operating with base three, 21011 becomes 20211 after three OSO's. Whereas, operating with base five, 21011 is converted into a member of the regenerative loop by two OSO's.

The index 00 belongs to the trivial *aaaaa* which has been excluded from the discussion, since it obviously vanishes upon one OSO.

Δ, *m* tabulations in other bases. As the size of the base increases, the flow charts become more complex. Consequently, only the Δ, m tabulations are given for other bases. They contain sufficient information to reconstruct the flow charts if desired. For bases where the terminal situation is not unanimous, the Δ's are tabulated in separate compartments headed by the specific corresponding self-producing integer or by loop members in order of their formation.

In base two, only one OSO is required to convert each of the two reps into one of the self-producing integers, 01111 or 10101.

Base Four

		20322 → 23331		
m		Δ		f
4	11	31	33	3
3	32			1
2	10	20	22	3
1	21	30		2
				9

Base Six

			41532			31533 → 35552		
m			Δ				Δ	f
6	10	33						2
5	20	31	50					3
4	11	21	41	51	55			5
3	52	54				53		3
2	22	44				30	43	4
1	42					32	40	3
								20

Base Seven

42633 → 40653 → 61641 → 54612 → 52632

m				Λ				*f*
6	10	30	50					3
5	20	32	42	60				4
4	21	51						2
3	11	40	61	63	64	66		6
2	22	31	33	44	52	54	65	7
1	41	43	53	55	62			5
								27

Base Eight

47773 → 42744 63732 → 52743 → 51753 → 61752

m		Δ					Δ				*f*
8			31	51							2
7			73	75							2
6			63	65							2
5			10	44	54						3
4			11	20	41	42	70	71	77		7
3			21	61	72	76					4
2	40		22	30	32	33	55	60	66	74	9
1	43	50	52	53	62	64					6
											35

Base Nine

62853 74832 → 63843 → 52854 → 60873 → 83841

m		Δ						Δ					*f*
7			50										1
6			44	54	55								3
5			10	30	41	51	70						5
4			11	20	32	62	80	81	83	86	88		9
3			21	42	52	71	73	76	82	87			8
2	33	66	22	31	40	43	60	65	72	74	77	85	12
1	63		53	61	64	75	84						6
													44

Base Ten

63954 → 61974 → 82962 → 75933

74943 → 62964 → 71973 → 83952

59994 → 53955

m			Δ			Δ				Δ	f
6	53				10						2
5	31	55	71		11	20	65	90			
						91	99				9
4	51	95			21	41	52	61			
						81	92	98			9
3	33	73	77	85	22	82	88	94			
		93	97			96					11
2	42	44	66	74	30	32	40	43			
		87				70	80	86	50		13
1	62	64	76	83	63	72	75	84	54	60	10
											54

In bases eleven and twelve, the symbols X and E represent the "digits" ten and eleven, respectively.

Base Eleven

The greatest number of steps to reach the unanimous loop

$$72X74 \rightarrow 82X73 \rightarrow 84X53 \rightarrow 73X64$$

is 8. This is required by integers with the indices 52, 62, $X3$, and $X8$.

Base Twelve

The greatest number of steps to reach the self-producing integer $83E74$ is the 9 required by integers with index 76.

Only 2 steps are required by integers with index 60 to reach the loop $6EEE5 \rightarrow 64E66$.

6 steps are needed to reach the loop

$$96E43 \rightarrow 84E64 \rightarrow 76E45 \rightarrow 71E95 \rightarrow X3E72$$

from integers with indices 10 and 64.

An abbreviated procedure. If predictive indices are not desired, the nature of the terminal situation and the OSO's necessary to reach it can be determined more rapidly. Consider the reversal-subtraction performed on the general rep:

$$\overline{r-1}\;\overline{r-1}\;\overline{r-1}\,d\,e - e\,d\;\overline{r-1}\;\overline{r-1}\;\overline{r-1}$$
$$= \overline{r-1-e}\;\overline{r-2-d}\;\overline{r-1}\,d\,\overline{e+1} = N_1.$$

In N_1, the sum of the extreme digits is r, the sum of the digits adjacent to them is $r-2$, and the central digit is $\overline{r-1}$. The sum of the five digits is $3(r-1)$. This holds except when $d=\overline{r-1}$. Then $N_1=\overline{r-2-e}\;\overline{r-1}\;\overline{r-1}\;\overline{r-1}\;\overline{e+1}$ in which the sum of the extremes is $r-1$ and the sum of the five digits is $4(r-1)$. In either case $\overline{e+1}>0$.

Thus to examine the entire field of five-digit integers (not of the form *aaaaa*) it suffices to apply the OSO only to those multiples of $r-1$ which meet the restrictions established in the previous paragraph. When these multiples are ordered, there are $(r/2)^2+(r/2-1)$ or $(r^2+2r-4)/4$ distinct integers when r is *even*, and $[(r-1)/2]^2+(r-1)/2$ or $(r^2-1)/4$ distinct integers when r is *odd*.

When an OSO is applied to each of these integers and the results are assembled into a flow chart, the terminal situation is revealed and the number of OSO's required to reach it from any location can be read off directly. Of course, one more OSO is required to move from an integer which generated a particular multiple than is required to move from that multiple to the terminal situation.

For example, the array involving the twenty reps in base six can be replaced by the following flow charts in which the eleven qualified multiples of five lead to a regenerative loop and to a self-producing integer.

```
53322
22335
-----
30543


54330      55541
03345      14555
-----      -----
50541      40542


55410      54420                      55221
01455      02445                      12255
----------------                      -----
55511      51531                      42522


55311                       55532     54222
11355                       23555     22245
-----                       ---------------
43512                      ┌─31533     31533
                           │
                           │
54321                      │ 53331
12345                      │ 13335
-----                      │ -----
41532–self-producing       └─35552–regenerative loop
```

Clearly, the greatest number of OSO's required to reach a member of a terminal situation will be the six required by one of the generators of 53322, say 41214 (equivalent to 44211 and to 55522). Thus $55522 - 22555 = 32523$ which reorders into 53322.

Some generalizations.

1. In systems with bases of the form $3k$, an OSO performed on the integer $\overline{2k}\ \overline{k-1}\ \overline{3k-1}\ \overline{2k-1}\ \overline{k}$ is:

$$\begin{array}{ccccc} \overline{3k-1} & \overline{2k} & \overline{2k-1} & \overline{k} & \overline{k-1} \\ \overline{k-1} & \overline{k} & \overline{2k-1} & \overline{2k} & \overline{3k-1}. \\ \hline \overline{2k} & \overline{k-1} & \overline{3k-1} & \overline{2k-1} & \overline{k} \end{array}$$

Thus in every system with base $r = 3k$, the integer $\overline{2r/3}\ \overline{r/3-1}\ \overline{r-1}\ \overline{2r/3-1}\ \overline{r/3}$ is self-producing. In this integer, $abcde$, $a+e=r$, $c=r-1$, and $b+d=r-2$.

2. In systems with bases of the form $2k$, with $k>1$, two successive OSO's performed on the integer $\overline{k}\ \overline{2k-1}\ \overline{2k-1}\ \overline{2k-1}\ \overline{k-1}$ are:

$$\begin{array}{ccccc} \overline{2k-1} & \overline{2k-1} & \overline{2k-1} & \overline{k} & \overline{k-1} \\ \overline{k-1} & \overline{k} & \overline{2k-1} & \overline{2k-1} & \overline{2k-1} \\ \hline \overline{k} & \overline{k-2} & \overline{2k-1} & \overline{k} & \overline{k} \end{array} \quad \text{and} \quad \begin{array}{ccccc} \overline{2k-1} & \overline{k} & \overline{k} & \overline{k} & \overline{k-2} \\ \overline{k-2} & \overline{k} & \overline{k} & \overline{k} & \overline{2k-1} \\ \hline \overline{k} & \overline{2k-1} & \overline{2k-1} & \overline{2k-1} & \overline{k-1}. \end{array}$$

Thus in every system with an even base $r>2$ there is a two-element regenerative loop:

$$\overline{r/2}\ \ \overline{r-1}\ \ \overline{r-1}\ \ \overline{r-1}\ \ \overline{r/2-1},\qquad \overline{r/2}\ \ \overline{r/2-2}\ \ \overline{r-1}\ \ \overline{r/2}\ \ \overline{r/2}.$$

In the second of these integers, $a+e=r$, $c=r-1$, and $b+d=r-2$. In the first element, $a+e=c=r-1$ and $b+d=2(r-1)$.

3. In systems with bases of the form $3k+2$ where $k>0$, four successive OSO's performed on the integer

$\overline{2k+1}\ \overline{k-1}\ \overline{3k+1}\ \overline{2k+1}\ \overline{k+1}$ are:

$$\begin{array}{ccccc} \overline{3k+1} & \overline{2k+1} & \overline{2k+1} & \overline{k+1} & \overline{k-1} \\ \overline{k-1} & \overline{k+1} & \overline{2k+1} & \overline{2k+1} & \overline{3k+1} \\ \hline \overline{2k+2} & \overline{k-1} & \overline{3k+1} & \overline{2k+1} & \overline{k} \end{array}, \quad \begin{array}{ccccc} \overline{3k+1} & \overline{2k+2} & \overline{2k+1} & \overline{k} & \overline{k-1} \\ \overline{k-1} & \overline{k} & \overline{2k+1} & \overline{2k+2} & \overline{3k+1} \\ \hline \overline{2k+2} & \overline{k+1} & \overline{3k+1} & \overline{2k+1} & \overline{k} \end{array},$$

$$\begin{array}{ccccc} \overline{3k+1} & \overline{2k+2} & \overline{2k+1} & \overline{k+1} & \overline{k} \\ \overline{k} & \overline{k+1} & \overline{2k+1} & \overline{2k+2} & \overline{3k+1} \\ \hline \overline{2k+1} & \overline{k} & \overline{3k+1} & \overline{2k} & \overline{k+1} \end{array}, \quad \begin{array}{ccccc} \overline{3k+1} & \overline{2k+1} & \overline{2k} & \overline{k+1} & \overline{k} \\ \overline{k} & \overline{k+1} & \overline{2k} & \overline{2k+1} & \overline{3k+1} \\ \hline \overline{2k+1} & \overline{k-1} & \overline{3k+1} & \overline{2k+1} & \overline{k+1} \end{array}.$$

Thus in every system having base $r = 3k+2$ with $k>0$, there is a four-element regenerative loop:

$$\overline{(2r-1)/3}\ \overline{(r-5)/3}\ \overline{r-1}\ \overline{(2r-1)/3}\ \overline{(r+1)/3},$$

$$\overline{(2r+2)/3}\ \overline{(r-5)/3}\ \overline{r-1}\ \overline{(2r-1)/3}\ \overline{(r-2)/3},$$

$$\overline{(2r+2)/3}\ \overline{(r+1)/3}\ \overline{r-1}\ \overline{(2r-7)/3}\ \overline{(r-2)/3},$$

$$\overline{(2r-1)/3}\ \overline{(r-2)/3}\ \overline{r-1}\ \overline{(2r-4)/3}\ \overline{(r+1)/3}.$$

In each integer of the loop, $a+e=r$, $c=r-1$, and $b+d=r-2$.

Summary.

1. All N_1's are divisible by $r-1$. The middle digit of every N_1 is $r-1$.
2. In the flow charts, an N at the level immediately preceding a self-producing integer is a permutation of the digits of that integer.
3. In base two there are two self-producing integers. Self-producing integers also occur in systems with bases of the form $3k$. The one in base three is unanimous. In bases six, nine and twelve, they are accompanied by loops. The one in base six, 41532, is a permutation of consecutive digits.
4. In systems having bases of the form $2k$, with $k>1$, there are two-element regenerative loops. In base four, the loop is unanimous.
5. In systems having bases of the form $3k+2$, with $k>0$, there are four-element regenerative loops. Those in bases five and eleven are unanimous. There are two four-element loops with base ten.
6. A unanimous five-element loop occurs with base seven. Five-element loops also occur with bases nine and twelve.
7. The prime bases five, seven, and eleven have unanimous loops.
8. In the cases studied, only the decimal system has three loops.
9. In one of the integers, *abcde*, in the two-element loops in base $2k$, with $k>1$, and in one of the self-producing integers in base two, $a+e=c=r-1$ and $b+d=2(r-1)$. In all other cases studied, in integers involved in terminal situations, $a+e=r$, $c=r-1$, and $b+d=r-2$.
10. In bases two through ten, tables are given which when entered with the predictive index of a five-digit integer, show the number, m, of OSO's necessary to go from the integer to a terminal situation when using Kaprekar's routine.

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THE TWO-TRIANGLE CASE OF THE ACQUAINTANCE GRAPH

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It was proved by Goodman [3] that at any 6-person party (where some people know each other and others do not), there is not only at least one triangle of mutual acquaintances or mutual strangers (that was already known [1],) but there are at least two such triangles!

The language of graph theory [4] is most convenient for handling such questions, and we use its terminology and notation here. The complete graph K_6 (Figure 1) has 6 points and 15 lines.

K_6:

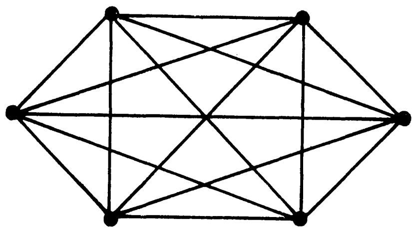


FIG. 1.

A *2-coloring* of K_6 is an assignment of either green or red to each line. If we draw a solid line for green and a dashed line for red, then Figure 2 shows a 2-coloring of K_6 with two disjoint green triangles and no red triangles. If solid and dashed lines are considered positive and negative, then we have a *signed graph* as studied in [5], Chapter 11.

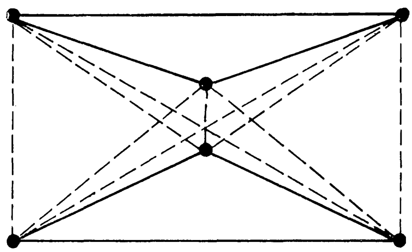


FIG. 2.

It is obvious that the maximum possible number of monochromatic triangles is obtained when all lines are green (say) and then there are $\binom{6}{3} = 20$ of them. But it is not trivial to specify which coloring patterns of the lines result in *exactly two* monochromatic triangles. It is our object to specify these patterns completely.

The shortest known proof that every 2-coloring of K_6 has *at least one* monochromatic triangle goes as follows. Every point u must be joined to at least three of the five other points by lines of the same color, say green as in Figure 3(a). Consider the three neighbors u_i of u. If there is even one green line joining a pair u_iu_j, we have a green triangle; if there is no such green line, then $u_1u_2u_3$ forms a red triangle as in Figure 3(b).

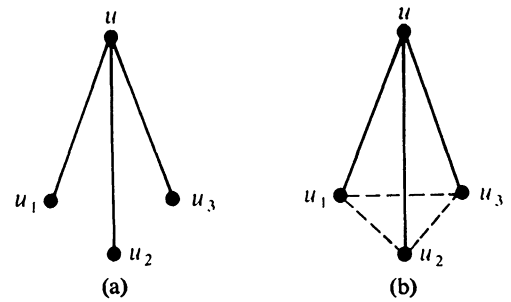


FIG. 3.

PREVIOUS THEOREM (Goodman [3]). *Every 2-coloring of K_6 has at least two monochromatic triangles.*

We now derive, by cases, all 2-colorings of K_6 (one of which is shown in Figure 2) having *exactly two* monochromatic triangles T_1 and T_2. Obviously T_1 and T_2 may intersect in 0, 1, or 2 points.

Case 0. *T_1 and T_2 have no common points.*

There are now two possibilities:

Case 0.1. *T_1 and T_2 have different colors.*

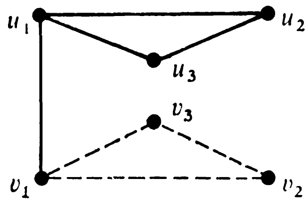


FIG. 4.

Label $u_1u_2u_3$ as a green triangle and $v_1v_2v_3$ as red. The line u_1v_1 must have a color; let it be green, as in Figure 4. Then line u_3v_1 must be red or $u_1u_3v_1$ would be a third monochromatic triangle. Similarly line u_3v_3 is next forced to be green, and u_2v_3 must be red as in Figure 5. But u_2v_1 cannot now be green (because u_1v_2 and u_1u_2 are green) or red (because of red lines u_2v_3 and v_1v_3) showing that Case 0.1 is impossible.

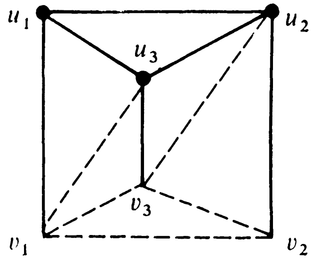


FIG. 5.

Case 0.2. *T_1 and T_2 have the same color*; let them both be green. Label

$T_1 = u_1u_2u_3$ and $T_2 = v_1v_2v_3$. The line u_1v_1 may be green, but then none of the other lines u_1v_j or u_iv_1 can be green or we have a third green triangle. Similarly u_2v_2 and u_3v_3 may be green. We summarize these observations:

When a 2-coloring of K_6 has exactly two monochromatic triangles $T_1 = u_1u_2u_3$ and $T_2 = v_1v_2v_3$ with no common point, they must both have the same color, say green, and then the three lines u_iv_i may be either green or red, but all the other lines must be red.

Figure 2 shows a 2-coloring of K_6 with T_1 and T_2 green and all other lines red, while Figure 6 shows T_1 and T_2 green and also the three lines u_iv_i green.

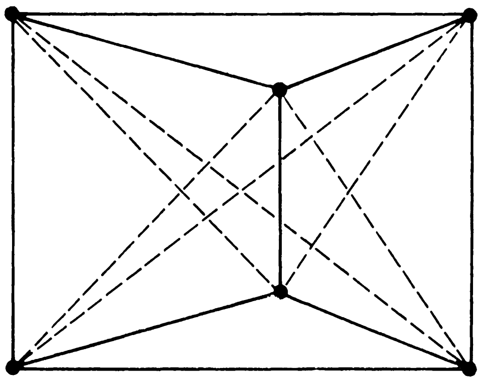


FIG. 6.

Case 1. *T_1 and T_2 have just one common point.*

We shall find that in this case, there only exist 2-colorings of K_6 such that T_1 and T_2 have different colors.

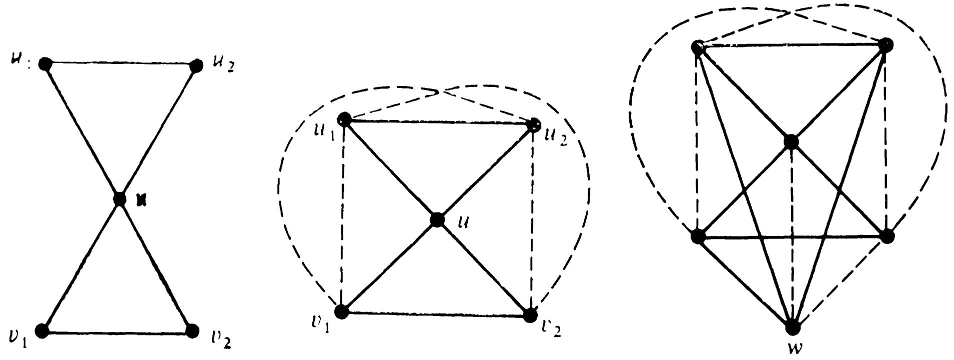
Case 1. 1. *T_1 and T_2 have the same color.*

Figure 7a shows two green triangles $T_1 = uu_1u_2$ and $T_2 = uv_1v_2$. In order to avoid a third green triangle, all four remaining lines on these five points must be red, as in Figure 7b. Now consider the sixth point w. Assume line wv_1 is red. Then both lines wu_1 and wu_2 must be green, causing a third green triangle wu_1u_2. Hence line wv_1 must be green, but then wu and wv_2 must both be red, which in turn forces both wu_1 and wu_2 to be green, as shown in Figure 7c. But now wu_1u_2 is again a third green triangle, so Case 1.1 is impossible!

Case 1.2. *T_1 and T_2 have different colors.*

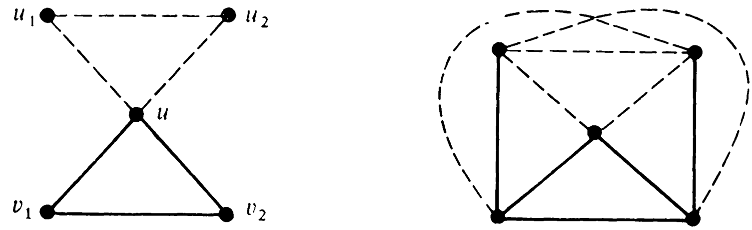
Let $T_1 = uu_1u_2$ be a red triangle and $T_2 = uv_1v_2$ be green, as in Figure 8a. Then the color of line u_1v_1 does not matter, so let it be green. In succession, as shown in Figure 8b, line u_1v_2 must be red, u_2v_2 green, and u_2v_1 red, thus completing the coloring of all 10 lines on these five points, and we find that there are five lines of each color.

Now consider the sixth point w. If we assume that wv_1 is green, then both wu and wu_1 must be red, so wuu_1 is a red triangle. Therefore, wv_1 is red. To avoid a third monochromatic triangle, we are forced to color in succession wu_2 green, wv_2



(a) (b) (c)

FIG. 7.



(a) (b)

FIG. 8.

red, and wu_1 green. At this stage, it actually does not matter whether the only remaining line wu is green or red; there will still be just the original two monochromatic triangles T_1 and T_2. Hence we draw wu in Figure 9 as a dotted line, to indicate this freedom of choice.

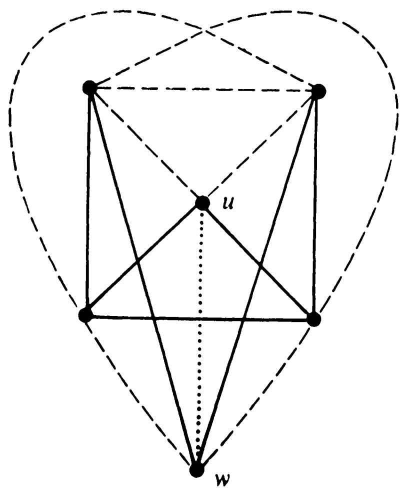


FIG. 9.

Case 2. *T_1 and T_2 share a line.*

Obviously both triangles must have the same color as the line they share, which we can take as green without loss of generality. The detailed analysis of the possible subcases of Case 2 proceeds quite similarly to those of Cases 0 and 1, and results in the unique 2-coloring of K_6 shown in Figure 10.

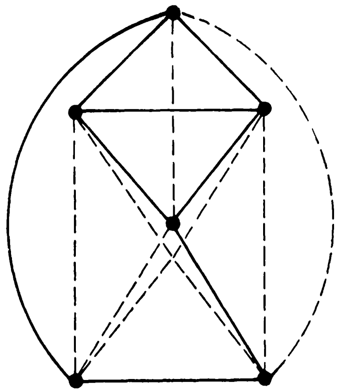


FIG. 10.

We now summarize these findings.

THEOREM. *There exist 2-colorings of K_6 with exactly two monochromatic triangles T_1 and T_2 which have* 0, 1, *or* 2 *common points. Further, T_1 and T_2 have different colors if and only if they have just one common point.*

All such 2-colorings of K_6 are indicated in Figures 2, 6, 9, and 10.

This problem was suggested by the ideas in our series of papers [2] on a generalized Ramsey theory for graphs. A host of other similar questions suggest themselves by taking other small subgraphs F with no isolates and deciding which 2-colorings of the complete graph K_r result in exactly the minimum possible number of monochromatic occurrences of F.

Underlying Theorem. When A. Schwenk studied a rough draft of this note, he noticed from the figures that in every 2-coloring of K_6 having the minimum number, two, of monochromatic triangles, the degrees of the points of each of the two monochromatic subgraphs G of K_6 were nearly equal. He succeeded in generalizing this observation to 2-colorings of any complete graph K_p with the minimum number of monochromatic triangles. This minimum number had already been determined exactly by Goodman:

THEOREM [3]. *If t is the number of monochromatic triangles in a 2-coloring of K_p, then*

$$t \geqq \binom{p}{3} - \left[\frac{p}{2}\left[\left(\frac{p-1}{2}\right)^2\right]\right].$$

By means of the following theorem, which was already implicit in Goodman's paper, Schwenk [6] obtained not only an alternate proof of Goodman's result, but also a method for deducing the explicit 2-colorings of K_6 in Figures 2, 6, 9, and 10 as corollaries, rather than by our exhaustive procedure.

As in the book [4], let $\delta(G)$ and $\Delta(G)$ denote respectively the minimum and maximum degrees of the points of graph G. Also, for x real we write $\{x\} = -[-x]$ as the smallest integer not less than x.

THEOREM [6]. *The bound for t in the preceding theorem is attained if and only if the degrees of each monochromatic subgraph G in a 2-coloring of* K_p *are as close to* $(p-1)/2$ *as possible, so that*

$$\left[\frac{p-1}{2}\right] \leqq \delta(G) \leqq \Delta(G) \leqq \left\{\frac{p-1}{2}\right\} \quad \textit{when} \quad p \not\equiv 3 \bmod 4,$$

and in the case $p \equiv 3 \bmod 4$, *G has just one point of degree* $(p-3)/2$ *or* $(p+1)/2$ *and all others have degree* $(p-1)/2$.

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1. C. W. Bostwick, E 1321, Amer. Math. Monthly, 66 (1959) 141–142.
2. V. Chvátal and F. Harary, Generalized Ramsey theory for graphs I, II, III, to appear.
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6. A. Schwenk, The acquaintance graph revisited, Amer. Math. Monthly, to appear.

MAXIMIZING THE SMALLEST TRIANGLE MADE BY N POINTS IN A SQUARE

MICHAEL GOLDBERG, Washington, D. C.

1. Introduction. Consider the placing of n points in a given region in the plane so that no three points are in a straight line. As a further strengthening of the misalignment, make the area of the smallest triangle as large as possible. If the given region is a circle, we may begin by placing the points along the circumference at the vertices of a regular n-gon.For small values of n, this arrangement may be the best. But for larger values of n, it is obvious that better arrangements are possible by placing some of the points in the interior of the circle.

If the given region is a rectangle, we may obtain an approximate solution by circumscribing the smallest parallelogram about the arrangement of n points in the circle. Then, an affine transformation carries the parallelogram into the shape of the given rectangle while the corresponding arrangement of the n points within the rec-

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If the given region is a rectangle, we may obtain an approximate solution by circumscribing the smallest parallelogram about the arrangement of n points in the circle. Then, an affine transformation carries the parallelogram into the shape of the given rectangle while the corresponding arrangement of the n points within the rectangle is produced. Since a continuous infinity of different parallelograms of equal area can circumscribe the set of points, there is a continuous infinity of equivalent arrangements for each arrangement. The affine transformation preserves the ratio of the area of each triangle to the area of the rectangle.

In this paper, we seek the best arrangement of n points in a square. Let the area of the smallest triangle be designated by $M(n)$. For each n, we shall try to find the largest value of $M(n)$.

2. Bounds on $M(n)$. Let the n points be the vertices of a regular polygon of n sides inscribed in a circle of unit diameter. Then the area $A(n)$ of the smallest triangle, whose vertices are three successive vertices of the regular n-gon, is given by

$$A(n) = (\tfrac{1}{4})\,(\sin 2\pi/n)^2 \tan \pi/n.$$

Circumscribe the smallest rectangle about the polygon. If n is even, then the longest sides of the rectangle contain a pair of opposite sides of the polygon. The height of this rectangle is $\cos \pi/n$. If $n = 4k$, then the width of the rectangle is also $\cos \pi/n$. If $n = 2k + 2$, then the width of the rectangle is unity. If we designate the ratio of the area of the smallest triangle to the area of the rectangle by $T(n)$, we obtain

$$M(n) \geqq T(n) = (\sin 2\pi/n)^2(\tan \pi/n)/(2 \cos \pi/n)^2, \quad \text{for } n = 4k;$$
$$M(n) \geqq T(n) = (\sin 2\pi/n)^2(\tan \pi/n)/4 \cos \pi/n, \quad \text{for } n = 2k + 2.$$

If $n = 2k + 1$, then the height of the circumscribed rectangle is $(\tfrac{1}{2})(1 + \cos \pi/n)$. The width of the rectangle is $\sin k\pi/n$. Then, a lower bound is given by

$$M(n) \geqq T(n) = (\sin 2\pi/n)^2\,(\tan \pi/n)/2\,(1 + \cos \pi/n) \sin k\pi/n,$$

for $n = 2k + 1$.

In any optimized arrangement of the points, at least one point must be placed on each side of the square. For the smaller values of n, all of these points will lie on the boundary of the square. However, for $n > 8$, there must be at least one point in the interior of the square; otherwise, there would be three points on a side of the square and, therefore, a triangle of zero area. From this interior point, draw lines through each of the other points. These lines divide the square into $n - 1$ regions. The smallest of these regions cannot exceed an area of $1/(n - 1)$. Since the smallest triangle cannot exceed this area, we have an upper bound given by $M(n) \leqq 1/(n - 1)$, for $n > 8$. This result was obtained as a solution of a problem posed by S. H. L. Kung [**1**]. A similar result was obtained by Roth [3, p.198] for the placing of n points in a triangle.

Thus, we have established upper and lower bounds on the value of $M(n)$.

3. Eight points and less. The largest triangle inscribed in a rectangle (or square) has an area equal to half of the rectangle. Hence, for three or four points in a square, the best value is half of the square. For less than eight points, the best arrangement seems to be the vertices of an affine regular polygon. These are shown in Figure 1, where one or more of the smallest triangles are drawn in each arrangement. Figure 2 shows two equivalent arrangements of six points in a square, and a less efficient ar-

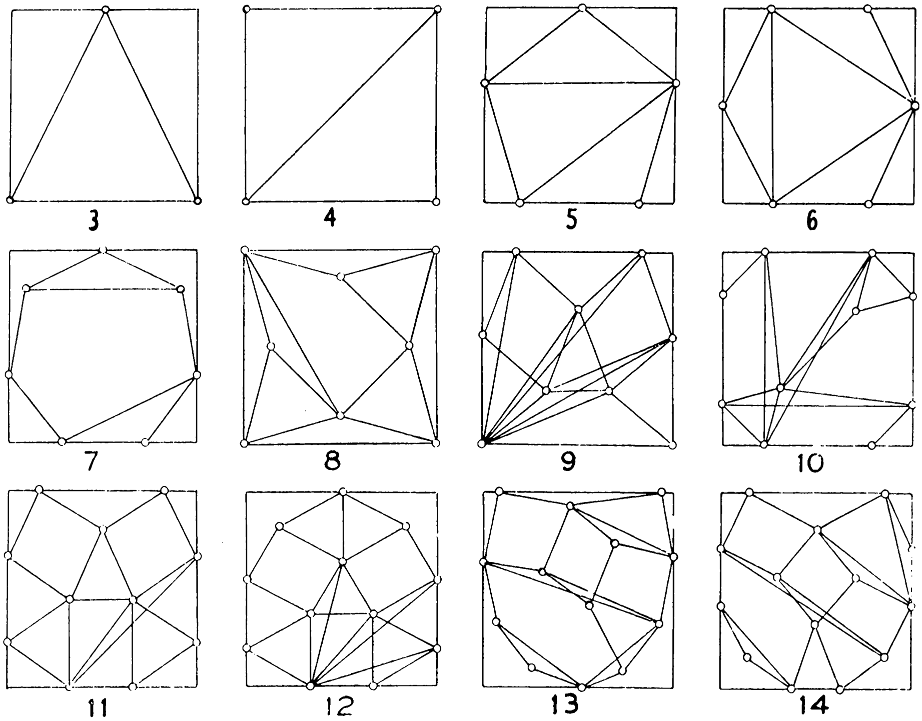
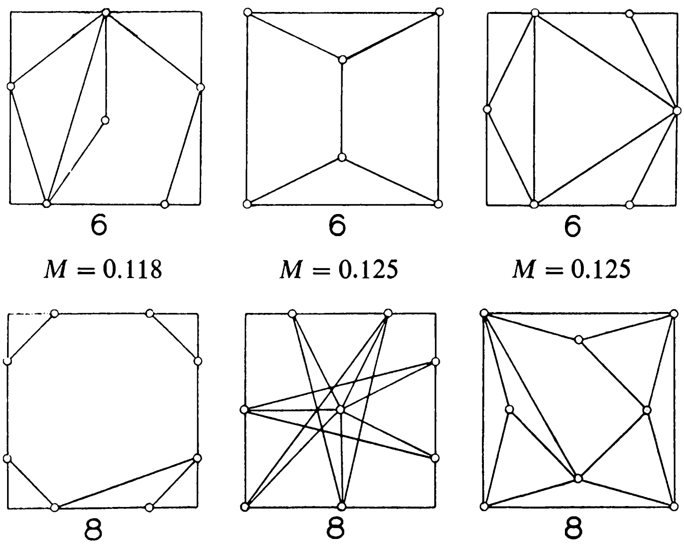


FIG. 1. Best arrangement of n points in a square.



$M = 0.118$ $M = 0.125$ $M = 0.125$

$M = 0.0625$ $M = 0.0625$ $M = 0.0670$

FIG. 2. Comparison of arrangements of six points and eight points.

rangement obtained by adding a point to the affine regular pentagon. For eight points, however, an improvement is made by the use of two different lengths for the sides of the octagon, as shown in Figure 2. If the points are at the distance x from the vertices of the square, then the area of the smallest triangle is given by $M = x(1-2x)/2$. This is maximized when $dM/dx = 0$, or $1/2 - 2x = 0$, from which $x = \frac{1}{4}$ and $M = 1/16 = 0.0625$. Another arrangement with the same value of M for eight points is shown in Figure 2. It was suggested by the best arrangement of seven circles in a square [**2**]. A still better arrangement, shown in Figures 1 and 2, is suggested by the best packing of eight circles in a square. The distance x of the interior points from the sides of the square is obtained by equating the areas of the small triangles. Then, $x = (2-\sqrt{3})/2$ and the area $M = (2-\sqrt{3})/4 = 0.0670$.

4. Ten points. The addition of two points to the semiregular octagonal of eight points of Figure 2, seems to give an efficient arrangement. If the points on the sides of the square are at the distance x from the vertices, and the interior points are at the distance $\sqrt{2}\,y$ from the corners, then by equating the areas of the small triangles we obtain $x = (5-\sqrt{17})/4$ and $y = (9-\sqrt{17})/16$, making the area of the smallest triangle equal to $(3\sqrt{17}-11)/32 = 0.0428$.

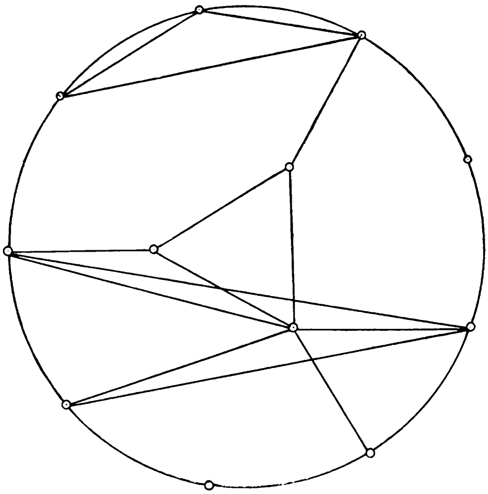


FIG. 3. Twelve points in a circle.

5. Twelve points. For a regular dodecagon inscribed in a circle of unit diameter, the area of the smallest triangle is

$$(\tfrac{1}{4})(\sin 2\pi/12)^2 \tan \pi/12 = (\tfrac{1}{4})(\sin 30^{0})^2 \tan 15^{0} = 0.017.$$

However, a more efficient arrangement of twelve points in a circle is shown in Figure 3, where nine of the points are at the vertices of a regular inscribed enneagon, while three points make an equilateral triangle in the interior. Then the area of the smallest triangle is 0.0242.

For twelve points in a rectangle the foregoing arrangement is modified in a symmetrical manner so that two more points lie on the boundary of the rectangle, as shown in Figure 4. This rectangle is nearly square. If this rectangle is compressed slightly in

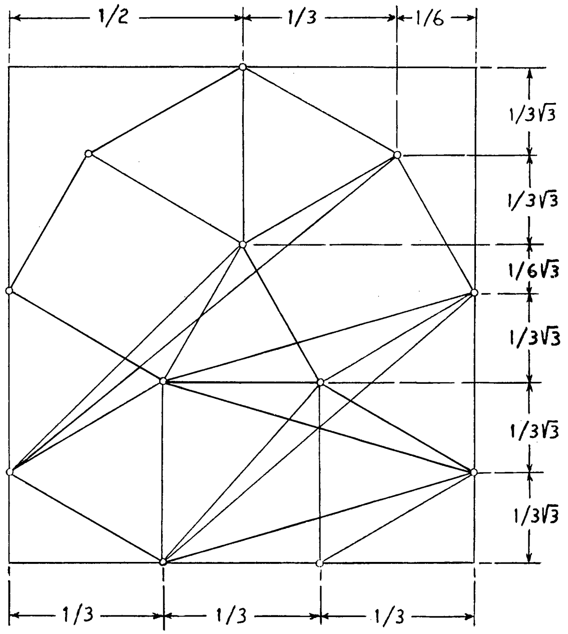


FIG. 4. Twelve points in a rectangle. Small triangle = Rectangle/33.

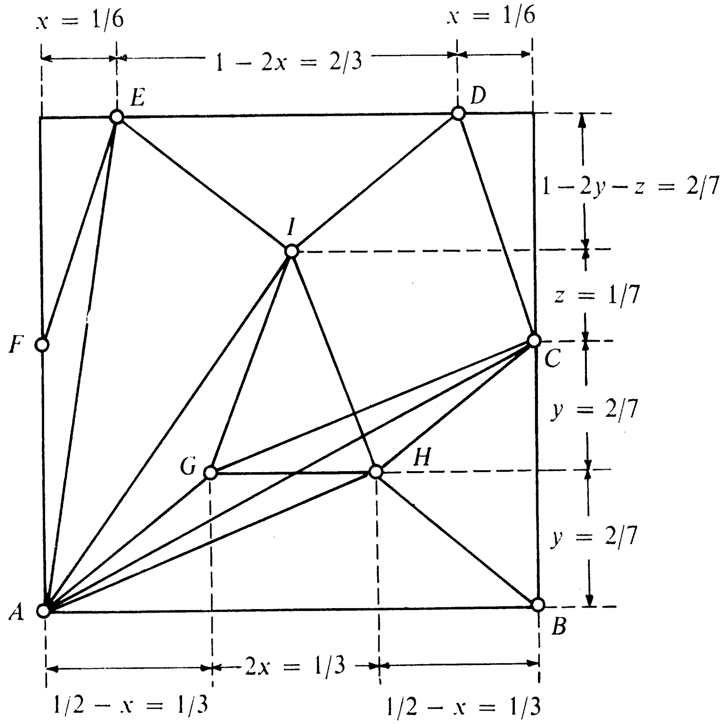


FIG. 5. Nine points in a square. Area of smallest triangle = 1/21 = 0.04762.

the vertical direction, it becomes a square, as shown in Figure 1. Since all areas are decreased in the same ratio, the ratio of the smallest area of a triangle to the area of the square remains $1/33 = 0.0303$.

6. Eleven points. For some values of n, good arrangements can be found by the omission of some of the points from highly efficient arrangements of slightly larger sets of points. For example, the omission of the top point of the arrangement of Figure 4 will give an arrangement of eleven points in a rectangle. If this rectangle is stretched to form a square, we obtain the arrangement of eleven points shown in Figure 1.

7. Nine points. The omission of the top point and the two bottom points of Figure 4 will give a good arrangement of nine points. This arrangement can be optimized as shown in Figure 5, where the nine points are shown as circled points labelled A to I. Let the distances x,y,z be unknown. Then, if the areas M of the small triangles are equated, we can solve for x,y,z.

$$\begin{aligned} \text{From}\quad M(AFE) &= M(AGH) = M(ACH) = xy \\ &= M(EIC) = (1-2y-2z+2xz)/4 \\ &= M(EIH) = (y+z-2x+2xy)/4 \\ &= M(AGI) = (y+z-4xy-2xz)/4, \end{aligned}$$

we obtain $x = 1/6$, $y = 2/7$, $z = 1/7$, and $M(9) = 1/21 = 0.04762$.

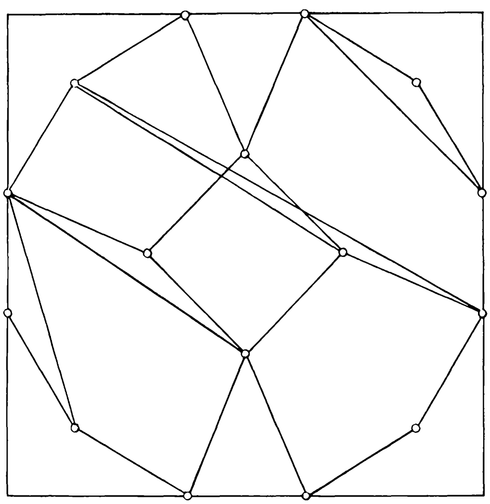


FIG. 6. Sixteen points in a square.

8. Fifteen points and sixteen points. An efficient arrangement of sixteen points, shown in Figure 6, consists of a semiregular polygon of twelve points within which is placed a small square of four points. Let the distance of the points on the sides of the large square from the corners be designated by y, and the distance of the other vertices of the dodecagon from the corners be designated by $\sqrt{2}z$. Let the four interior points be at the distance x from the center. Then by equating the areas of the four small shallow triangles shown in Figure 6, we obtain $x = 0.207$, $y = 0.376$, $z = 0.141$ and $M(16) = 0.0175$.

No arrangement of fifteen points has been found that is better than the omission of a point from the foregoing arrangement of sixteen points.

9. Thirteen points and fourteen points. As in the case of eleven points, a good arrangement of fourteen points can be obtained by omitting two points from the arrangement of sixteen points and stretching the remainder to fill the square. Similarly, an arrangement of thirteen points can be obtained by omitting three points. It is likely that one or both of these arrangements can be improved by adjustments of the remaining points. These arrangements are shown in Figure 1.

10. Summary. The best area obtained for each n is shown in the following table. Also, n^2 times the area is tabulated. The higher the latter figure, the more efficient is the arrangement. It should be noted that an approximate value for the area is given by $4/n^2$.

TABLE OF AREAS

n	Area			n^2 Area
3	1/2	=	0.500	4.500
4	1/2	=	0.500	8.000
5	$(3-\sqrt{5})/4$	=	0.191	4.750
6	1/8	=	0.125	4.500
7		=	0.079	3.92
8	$(2-\sqrt{3})/4$	=	0.067	4.288
9	1/21	=	0.048	3.85
10	$(3\sqrt{17}-11)/32$	=	0.043	4.28
11	1/27	=	0.037	4.48
12	1/33	=	0.030	4.36
13		=	0.022	3.78
14		=	0.020	4.00
15				
16		=	0.0175	4.48

11. The median condition. In every optimized arrangement, there are many smallest triangles of the same area. This is a necessary condition; otherwise, some of the points could be moved to increase the area of the smallest triangle at the expense of a larger triangle. However, this is not a sufficient condition for the best arrangement, as is illustrated by the comparison of the three optimized arrangements of eight points shown in Figure 2.

If an interior point P is a vertex of two small triangles PQR and PQS of the same area, then P is on the median of the triangle QRS. If the three triangles PQR, PQS, PRS have equal areas, then P is the centroid of the triangle QRS. As an example, the interior points of the arrangement of ten points, shown in Figure 1, are the centroids of such triangles.

12. An analytical formulation of the problem. The following remarks constitute an attempt at reducing the problem of the points in a square to an algebraic or analytical problem. If (x_1,y_1), (x_2,y_2), (x_3,y_3) are the coordinates of the vertices of a triangle, then the area of the triangle is given by $(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2)/2$. For n points, there are $n(n-1)(n-2)/6$ triangles. These are scanned to find the smallest triangle. The points are varied to maximize the smallest triangle. The solution is the location of these points.

It should be noted that the solution is not always unique. In most cases, some of the points may still be varied without changing the area of the smallest triangle. In particular, these may be some of the boundary points which are the vertices of only the larger triangles.

The intial problem may be modified by the use of criteria which retain some of the larger triangles while retaining the area of the smallest triangle. There are many ways of choosing such criteria. One of these may be the use of a "figure of merit" which is the product of the area of all the triangles. Then, this product is maximized. Thus, if the small triangles are preserved, this product is increased when some of the larger triangles are subsequently increased. Furthermore, this excludes solutions in which the small triangles are reduced excessively. It does, however, admit the possibility of some reduction in the area of the smallest triangle to enable a very large increase in the area of some of the larger triangles. It is to be expected, however, that the arrangement so obtained is a good approximation of the arrangement which maximizes only the smallest triangle.

If the area of a triangle is represented by $\{z_i,z_j,z_k\}$, then $F(n)$, the figure of merit for n points, may be expressed as

$$F(n) = \max \prod^{n} \{z_i,z_j,z_k\}, \quad \text{for} \quad i \neq j,\ i \neq k,\ j \neq k.$$

13. Large numbers of points. As n is increased, it is expected that for the best arrangements, the swarms of points will break down into approximations of concentric rings. No method of determining these arrangements has been found.

For large values of n, one may consider close packing of n equal circles in a square. The area of the equilateral triangle made by the centers of three contacting circles is approximately $1/2n$. Hence, this serves as a closer upper bound, since there are always triplets of centers which make smaller triangles.

Also, based on a suggestion by Dr. R. L. Graham of the Bell Telephone Laboratories, a lower bound that is closer than the regular polygon bound, is obtained when n is very large. For n equal to a prime p, divide the square into a lattice of $(n-1)^2$ squares. Place the points at the lattice points whose coordinates are the integers (ak, k^2) mod n. Then, no three of the points are collinear. The smallest triangle is never less than $1/2(n-1)^2$ in area. For most cases, the smallest triangle may have twice this area, namely $1/(n-1)^2$. For examples, the following distributions yield the latter area:- $(2k, k^2)$ mod 11, $(4k, k^2)$ mod 13, (k, k^2) mod 17, $(4k, k^2)$ mod 19, $(2k, k^2)$ mod 29, $(6k, k^2)$ mod 31.

When p is of the form $4m+3$, a slightly larger area is obtained. For example,

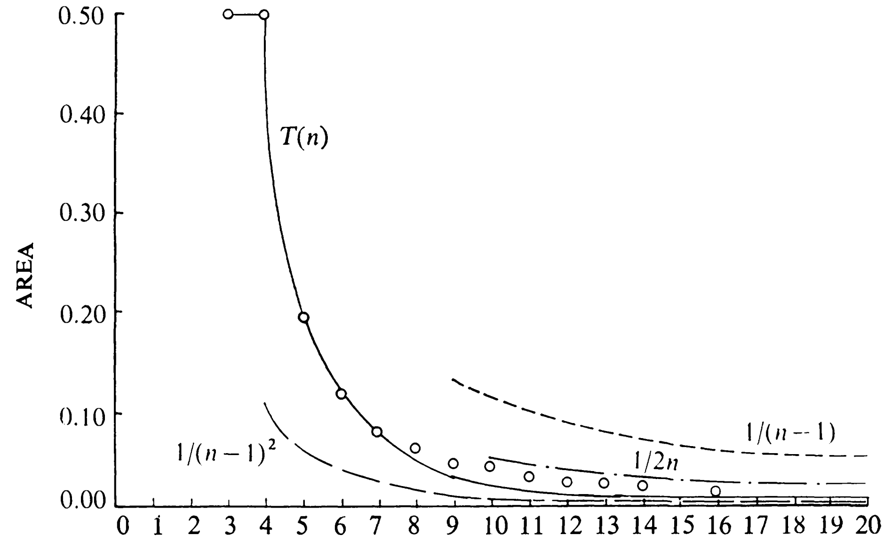


FIG. 7. Graph of areas and bounds.

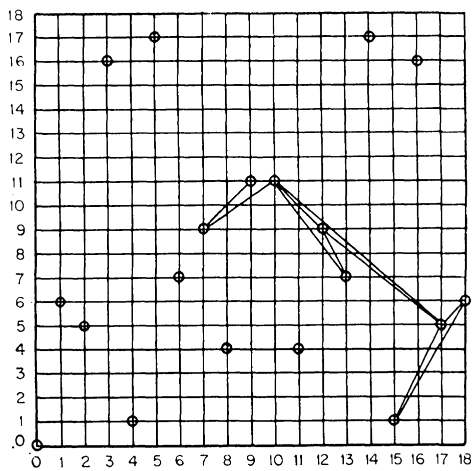


FIG. 8. Lattice arrangement of nineteen points in a square. $(4K, K^2)$ mod 19.

when $n = 19$, a rectangle which is smaller than the square will enclose all the points. Then the rectangle can be stretched to make it into a square with a resultant increase in the area of the smallest triangle. For $n = 19$, it becomes $1/(17 \cdot 18) > 1/18^2$.

In each of the foregoing arrangements, the elimination of the point at (0, 0) produces an arrangement of $(p - 1)$ points in a square of edge $(p - 2)$. Hence, for $n = p - 1$, the area of the smallest triangle is again $1/(n - 1)^2$.

A similar bound was found by Roth [3, p. 204] for the problem of placing n points in a triangle.

In the graph of Figure 7, the best values are designated by the encircled points.

The upper bound $1/(n-1)$ is shown as a dotted curve. The upper bound $1/2n$ is shown as a dash-dot curve. The lower bound of the regular polygons is shown as a solid curve. The lower bound for the lattice arrangements is shown as the dashed curve. Although definite lattice arrangements are available only for the primes p and for $p-1$, the latter curve can serve as an approximate bound for the other values of n, since the function must be nonincreasing.

For large n, the area $T(n)$ of the small triangle based on the regular polygon arrangement in a square is given by

$$T(n) \sim (2\pi/n)^2(\pi/n)/4 = (\pi/n)^3.$$

Hence, if $(\pi/n)^3 = 1/(n-1)^2$, we obtain $n \approx 28$. Therefore, for $n > 28$, the lattice arrangement gives a greater least area than the regular polygon arrangement.

14. Generalizations. This problem can be extended in various ways. Instead of a square region, one may use other polygons or even other closed curves. Also, it can be generalized to three dimensional regions with the requirement that the smallest tetrahedron be maximized. A similar upper bound can be established by the consideration of the close packing of spheres. Also, a similar lower bound, for large n, can be obtained by the lattice arrangement. The regular polyhedron bound is not generally available because there are only five regular polyhedra. However, it is expected that the multisymmetric polyhedra may serve to establish an approximate lower bound.

References

1. Problem No. 745 (proposed by S. H. L. Kung), this MAGAZINE, 43 (1970) 170–171.
2. Michael Goldberg, The packing of equal circles in a square, this MAGAZINE, 43 (1970) 24–30.
3. K. F. Roth, On a problem of Heilbronn, J. London Math. Soc., 26 (1951) 198–204.

FACTORABLE DETERMINANTS

K. O. BOWMAN, Oak Ridge National Laboratory, Tennessee,
and L. R. SHENTON, University of Georgia

In a paper [1] published in 1858, Painvin showed that the determinant

$$\begin{vmatrix} r & 1/2 & 0 & 0 & & \\ -n/2 & r-1 & 2/2 & 0 & & \\ 0 & -(n-1)/2 & r-2 & 3/2 & & \\ 0 & 0 & -(n-2)/2 & r-3 & & \\ & & & -3/2 & r-n+1 & n/2 \\ & & & & -1/2 & r-n \end{vmatrix}$$

The upper bound $1/(n-1)$ is shown as a dotted curve. The upper bound $1/2n$ is shown as a dash-dot curve. The lower bound of the regular polygons is shown as a solid curve. The lower bound for the lattice arrangements is shown as the dashed curve. Although definite lattice arrangements are available only for the primes p and for $p-1$, the latter curve can serve as an approximate bound for the other values of n, since the function must be nonincreasing.

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of order $n+1$ has the value $(r-\frac{1}{2}n)^{n+1}$. This determinant can thus be expressed as the product of $n+1$ determinants of order 1 each having $r-\frac{1}{2}n$ as its single element, and the determinant can thus be considered to be factorable. We consider here two other factorable determinants.

A modest amount of computation shows that

$$\begin{vmatrix} n-4 & 0 & 4 \\ 3 & n-2 & -4 \\ -1/2 & 1 & 1 \end{vmatrix} = n^2 \tag{1}$$

$$\begin{vmatrix} n-6 & 0 & 0 & -8 \\ 5 & n-4 & 0 & 12 \\ -2/2 & 3 & n-2 & -6 \\ 0 & -1/2 & 1 & 1 \end{vmatrix} = n^3, \tag{2}$$

$$\begin{vmatrix} n-8 & 0 & 0 & 0 & 16 \\ 7 & n-6 & 0 & 0 & -32 \\ -3/2 & 5 & n-4 & 0 & 24 \\ 0 & -2/2 & 3 & n-2 & -8 \\ 0 & 0 & -1/2 & 1 & 1 \end{vmatrix} = n^4. \tag{3}$$

The generalization to a determinant of order $s+1$ is that

$$|A_s(n)| = n^s, \quad \text{where} \tag{4}$$

$$A_s(n) = \begin{bmatrix} n-2s & 0 & \cdots & 0 & 0 & 0 & (-2)^s \\ 2s-1 & n-2s+2 & \cdots & \cdot & \cdot & \cdot & \binom{s}{1}(-2)^{s-1} \\ (1-s)/2 & 2s-3 & \cdots & \cdot & \cdot & \cdot & \binom{s}{2}(-2)^{s-2} \\ 0 & (2-s)/2 & \cdots & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdots & n-6 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdots & 5 & n-4 & 0 & \cdot \\ \cdot & \cdot & \cdots & -2/2 & 3 & n-2 & \cdot \\ 0 & 0 & \cdots & 0 & -1/2 & 1 & \binom{s}{s}(-2)^{s-s} \end{bmatrix} \tag{5}$$

$(s = 1, 2, \cdots)$.

The determinant would be simple to evaluate if the last column consisted of zeros. However, the matrix can be evaluated by what amounts to successive applications of premultiplication by $H_{s+1}(r)$ and postmultiplication by $K_{s+1}(r)$ $(r = 0, 1, \cdots, s-1)$, where

$$H_{s+1}(r) = \begin{bmatrix} I_r & O \\ O' & B_{s-r+1} \end{bmatrix} \tag{6a}$$

with I_r an $r \times r$ unit matrix, B_{s-r+1} an upper triangular matrix with elements 2^m along the diagonal through $(1, m+1)$ with $m = 0, 1, \cdots, s-r$; moreover

$$K_{s+1}(r) = \begin{bmatrix} H_s^{-1}(r) & O \\ O' & 1 \end{bmatrix} \tag{6b}$$

where O is a null column matrix with s rows, O' its transpose.

Now it may be verified, after using the binomial identities

$$\binom{s}{1} - \binom{s}{2} + \cdots + (-1)^{s-1}\binom{s}{s} = 1, \tag{7a}$$

$$\binom{s}{2} - \binom{s}{3} + \cdots + (-1)^{s-2}\binom{s}{s} = \binom{s-1}{1}, \tag{7b}$$

and more generally

$$\binom{s}{r} - \binom{s}{r+1} + \cdots + (-1)^{s-r}\binom{s}{s} = \binom{s-1}{r-1} \qquad (r = 1, 2, \cdots, s;\ s = 2, 3, \cdots), \tag{7c}$$

that if

$$A_s^{(0)}(n) = H_{s+1}(0)\, A_s(n) K_{s+1}(0), \tag{8a}$$

then $A_s^{(0)}(n)$ is a matrix of order $s+1$ with elements n, $n-2s$, $n-2s+2$, $n-2s+4, \cdots n-6, n-4, 1$ in the main diagonal; s, $2s-2, 2s-4, \cdots, 6, 4, 2$ in the first subdiagonal; $(1-s)/2$, $(2-s)/2, \cdots, -2/2, -1/2$ in the second subdiagonal; 0 followed by the binomial terms $(-2)^{s-1}$, $\binom{s-1}{1}(-2)^{s-2}$, $\binom{s-1}{2}(-2)^{s-3}, \cdots$, $\binom{s-1}{s-1}(-2)^0$ in the last column, and zeros elsewhere (note the last element of the last column is unity, agreeing with the last element of the main diagonal). Similarly if

$$A_s^{(1)}(n) = H_{s+1}(1) A_s^{(0)}(n) K_{s+1}(1) \tag{8b}$$

then $A_s^{(1)}(n)$ has elements n, n, $n-2s$, $n-2s+2, \cdots, n-8, n-6, 1$ in the main diagonal; 1, s, $2s-3$, $2s-5, \cdots, 5, 3$ in the first subdiagonal; $(1-s)/2, (2-s)/2, \cdots, -\frac{1}{2}$ in the second subdiagonal; and last column 0, 0 followed by the binomial terms $(-2)^{s-2}$, $\binom{s-2}{1}(-2)^{s-3}$, $\binom{s-2}{2}(-2)^{s-4}, \cdots, 1$.

Finally, one finds

$$H_{s+1}(s-1) H_{s+1}(s-2) \cdots H_{s+1}(0) A_s(n) K_{s+1}(0) \cdots K_{s+1}(s-2) K_{s+1}(s-1) =$$

$$
(9)\qquad
\begin{bmatrix}
n & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & n & 0 & \cdots & \cdot & \cdot & \cdot \\
(1-s)/2 & 2 & n & \cdots & \cdot & \cdot & \cdot \\
\cdot & (2-s)/2 & 3 & \cdots & \cdot & \cdot & \cdot \\
\cdot & \cdot & (3-s)/2 & \cdots & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & n & 0 & \cdot \\
\cdot & \cdot & \cdot & \cdots & s-1 & n & 0 \\
\cdot & \cdot & \cdot & \cdots & -\tfrac{1}{2} & s & 1
\end{bmatrix}
$$

from which (4) follows.

There is also a similar result for determinants with even elements in the first subdiagonal. Thus

$$
(10)\qquad
\begin{vmatrix}
n-2s+2 & 0 & 0 & \cdots & 0 & 0 & 0 & (-2)^s \\
2s-2 & n-2s+4 & 0 & \cdots & \cdot & \cdot & \cdot & \binom{s}{1}(-2)^{s-1} \\
(1-s)/2 & 2s-4 & n-2s+6 & \cdots & \cdot & \cdot & \cdot & \cdot \\
0 & (2-s)/2 & 2s-6 & \cdots & \cdot & \cdot & \cdot & \cdot \\
\cdot & 0 & (3-s)/2 & \cdots & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & n-4 & 0 & 0 & \cdot \\
\cdot & \cdot & \cdot & \cdots & 4 & n-2 & 0 & \cdot \\
\cdot & \cdot & \cdot & \cdots & -2/2 & 2 & n & \cdot \\
0 & 0 & 0 & \cdots & 0 & -1/2 & 0 & \binom{s}{s}(-2)^0
\end{vmatrix}_{(s+1)}
$$

$$
= n^s \qquad (s = 1, 2, \cdots).
$$

One use for these results is testing out computer programs for evaluating determinants of large order, especially if there is interest in error assessment.

Acknowledgment: We are very much indebted to a referee who pointed out an error in our initial proof.

This research was sponsored by the U. S. Atomic Energy Commission under contract with Union Carbide Corporation.

Reference

1. L. Painvin, Sur un certain système d'équations linéaires, J. Math. Pures Appl., 2 (1858) 41–46.

STILL ANOTHER ELEMENTARY PROOF THAT $\Sigma\, 1/k^2 = \pi^2/6$

DANIEL P. GIESY, Western Michigan University

This well-known formula, usually proven by appealing to some nonelementary results of Fourier series or complex analysis, for example, is established in [1] and in [2] by comparatively elementary means. We give another elementary proof which we feel has some advantages over those of [1] and [2].

If we set

$$f_n(x) = \tfrac{1}{2} + \cos x + \cdots + \cos nx \tag{1}$$

then it is easily verified by induction and routine trigonometric identities that

$$f_n(x) = \frac{\sin[(2n+1)x/2]}{\sin(x/2)} \tag{2}$$

then using (1) for f_n, we find

$$E_n = \int_0^\pi x f_n(x)\,dx = \frac{\pi^2}{4} + \sum_{k=1}^{n} \frac{(-1)^k}{k^2} - \frac{1}{k^2}$$

so that

$$\tfrac{1}{2} E_{2n-1} = \frac{\pi^2}{8} - \sum_{k=1}^{n} \frac{1}{(2k-1)^2}. \tag{3}$$

On the other hand, using (2) for f_n, setting

$$g(x) = \frac{d}{dx} \frac{x/2}{\sin(x/2)}$$

and integrating by parts, we find

$$E_{2n-1} = \left(2 + 2\int_0^\pi g(x)\cos[(4n-1)x/2]\,dx\right)/(4n-1)$$

where we have used L'Hospital's rule to see that $\lim_{x\to 0}(x/2)/\sin(x/2) = 1$. Further use of L'Hospital's rule establishes that $g(0) = 0 = \lim_{x\to 0} g(x)$ and investigation of g' shows that g is increasing on $[0,\pi]$; therefore, g is bounded on $[0,\pi]$ (by $g(\pi) = \frac{1}{2}$) and so $\lim_{n\to\infty} E_{2n-1} = 0$. Thus from (3) we see that

$$\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}.$$

By comparison to this series, $\Sigma_{k=1}^{\infty} 1/(2k)^2$ converges, say to L, and therefore

$$\sum_{k=1}^{\infty} 1/k^2 = 4\sum_{k=1}^{\infty} 1/(2k)^2 = 4L.$$

From $4L = \pi^2/8 + L$, we see that $L = \pi^2/24$, and finally,

$$\sum_{k=1}^{\infty} 1/k^2 = 4L = \pi^2/6.$$

Finally, we thank Paul Eenigenburg for a valuable suggestion.

References

1. Y. Matsuoka, An elementary proof of the formula $\Sigma_{k=1}^{\infty} 1/k^2 = \pi^2/6$, Amer. Math. Monthly, 68 (1961) 485–487.

2. E. L. Starke, Another proof of the formula $\Sigma\ 1/k^2 = \pi^2/6$, Amer. Math. Monthly, 76 (1969) 552–553.

A BOUNDARY VALUE PROBLEM

CLARENCE R. EDSTROM, Air Force Institute of Technology

We consider the boundary value problem

$$y_{tt} = a^2 y_{xx} \tag{1}$$

with

$$y(x,0) = y_t(x,0) = y(L,t) = 0,\ y(0,t) = A\sin\frac{a\pi t}{L}. \tag{2}$$

This may be interpreted as describing the motion of a string of length L fixed at one end and shaken at the other end.

Reddick and Miller [**1**, p. 398 and p. 518] give a numerical solution as well as a Laplace transform solution of this problem. A condition of resonance is shown to exist and the solution is determined to be unbounded with respect to time. There is a third standard method of attack on such problems which works well in this case.

We change the dependent variable $y(x,t)$ by letting

$$y(x,t) = Y(x,t) + A\left(1 - \frac{x}{L}\right)\cos\frac{\pi x}{L}\sin\frac{a\pi t}{L} - \frac{aAt}{L}\sin\frac{\pi x}{L}\cos\frac{a\pi t}{L}. \tag{3}$$

The boundary value problem becomes

$$Y_{tt} = a^2 Y_{xx} \tag{4}$$

with

$$\begin{cases} Y(0,t) = Y(L,t) = Y(x,0) = 0 \\ Y_t(x,0) = \dfrac{aA}{L}\sin\dfrac{\pi x}{L} + \dfrac{aA\pi(x-L)}{L^2}\cos\dfrac{\pi x}{L}. \end{cases} \tag{5}$$

These boundary conditions do not involve t so we can use the method of separation of variables. We obtain as the solution of this new problem

$$\sum_{k=1}^{\infty} 1/k^2 = 4L = \pi^2/6.$$

Finally, we thank Paul Eenigenburg for a valuable suggestion.

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These boundary conditions do not involve t so we can use the method of separation of variables. We obtain as the solution of this new problem

$$(6) \qquad Y(x,t) = \frac{A}{2\pi}\sin\frac{a\pi t}{L}\sin\frac{\pi x}{L} + \frac{2A}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n \sin(an\pi t/L)\sin(n\pi x/L)}{n^2-1}.$$

Thus the solution of the original problem is

$$y(x,t) = A\left(1-\frac{x}{L}\right)\cos\frac{\pi x}{L}\sin\frac{a\pi t}{L} - \frac{aAt}{L}\sin\frac{\pi x}{L}\cos\frac{a\pi t}{L}$$

$$(7) \qquad + \frac{A}{2\pi}\sin\frac{a\pi t}{L}\sin\frac{\pi x}{L} + \frac{2A}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n \sin(an\pi t/L)\sin(n\pi x/L)}{n^2-1}.$$

Evaluating (7) at $(L/2, 2kL/a)$ where $k = 1, 2, 3, \cdots$, we obtain

$$(8) \qquad y\left(\frac{L}{2}, \frac{2kL}{a}\right) = -2Ak.$$

Thus as $t \to \infty$, the displacement is not bounded.

Reference

1. H. W. Reddick and F. H. Miller, Advanced Mathematics for Engineers, 3rd ed., Wiley, New York, 1955.

PROJECTIVITIES IN $PG_{10(nd)}$

STEVEN ADAMSON and C. R. WYLIE, Furman University

It is well known that in classical projective geometry obtained, say, by adjoining the appropriate ideal elements to the euclidean plane, a projectivity between two lines is uniquely determined by the assignment of three pairs of corresponding points. This result is not a consequence of the usual axioms of incidence and connection for a projective geometry, however, and hence itself is a candidate for inclusion as an axiom. If it is assumed, then the perspectivity theorem and the theorems of Pappus and Desargues follow readily and, with the appropriate additional axioms, the classical projective plane is obtained. Contrapositively, in a projective geometry in which Desargues' theorem is false, the projectivity axiom cannot hold, and two projectivities may have more than three pairs of corresponding points in common.

Heath [1] undertook a limited investigation of this matter for one of the finite nondesarguesian geometries having 10 points per line and 91 points altogether, and found instances of pairs of projectivities having 0, 1, 2, — 6 pairs of corresponding points in common, but none with more than 6. To explore the matter further, we made a computer search, using facilities of the Furman University Computer Center to implement the following procedure: Two of the 91 lines in $PG_{10(nd)}$ were selected at random. Then two projectivities between these lines were constructed, each being the composition of two perspectivities for which the centers and the intermediate axis were also randomly selected.

$$(6)\qquad Y(x,t) = \frac{A}{2\pi}\sin\frac{a\pi t}{L}\sin\frac{\pi x}{L} + \frac{2A}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n\sin(an\pi t/L)\sin(n\pi x/L)}{n^2-1}.$$

Thus the solution of the original problem is

$$y(x,t) = A\left(1-\frac{x}{L}\right)\cos\frac{\pi x}{L}\sin\frac{a\pi t}{L} - \frac{aAt}{L}\sin\frac{\pi x}{L}\cos\frac{a\pi t}{L}$$

$$(7)\qquad + \frac{A}{2\pi}\sin\frac{a\pi t}{L}\sin\frac{\pi x}{L} + \frac{2A}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n\sin(an\pi t/L)\sin(n\pi x/L)}{n^2-1}.$$

Evaluating (7) at $(L/2, 2kL/a)$ where $k = 1, 2, 3, \cdots$, we obtain

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A total of 6000 runs was made, in which we found a single pair of projectivities with 7 pairs of corresponding points in common but none with more than 7. A limited amount of nonrandom searching led to a second pair of projectivities with 7 pairs of corresponding points in common but, again, none with more than 7. Examples of such pairs of projectivities for $n = 0, 1, \text{—} 7$ are listed in Table 1, the notation being that of the incidence tables for $PG_{10(nd)}$ appearing in [2].

TABLE 1

Number of pairs of mates in common	Projectivities	
0	$l_{13} \overset{P_7}{\overline{\wedge}} l_{41} \overset{P_8}{\overline{\wedge}} l_2$,	$l_{13} \overset{P_{70}}{\overline{\wedge}} l_{32} \overset{P_{44}}{\overline{\wedge}} l_2$
1	$l_{13} \overset{P_{70}}{\overline{\wedge}} l_{50} \overset{P_2}{\overline{\wedge}} l_2$,	$l_{13} \overset{P_{27}}{\overline{\wedge}} l_{41} \overset{P_{60}}{\overline{\wedge}} l_2$
2	$l_{13} \overset{P_7}{\overline{\wedge}} l_{32} \overset{P_6}{\overline{\wedge}} l_2$,	$l_{13} \overset{P_9}{\overline{\wedge}} l_{14} \overset{P_7}{\overline{\wedge}} l_2$
3	$l_{13} \overset{P_7}{\overline{\wedge}} l_{50} \overset{P_2}{\overline{\wedge}} l_2$,	$l_{13} \overset{P_{27}}{\overline{\wedge}} l_{50} \overset{P_{36}}{\overline{\wedge}} l_2$
4	$l_{13} \overset{P_7}{\overline{\wedge}} l_{50} \overset{P_2}{\overline{\wedge}} l_2$,	$l_{13} \overset{P_{27}}{\overline{\wedge}} l_{41} \overset{P_{60}}{\overline{\wedge}} l_2$
5	$l_{13} \overset{P_7}{\overline{\wedge}} l_{50} \overset{P_2}{\overline{\wedge}} l_2$,	$l_{13} \overset{P_{27}}{\overline{\wedge}} l_{68} \overset{P_{31}}{\overline{\wedge}} l_2$
6	$l_1 \overset{P_{12}}{\overline{\wedge}} l_{20} \overset{P_{15}}{\overline{\wedge}} l_{11}$,	$l_1 \overset{P_{13}}{\overline{\wedge}} l_{29} \overset{P_{27}}{\overline{\wedge}} l_{11}$
7	$l_{13} \overset{P_{27}}{\overline{\wedge}} l_{77} \overset{P_{50}}{\overline{\wedge}} l_2$,	$l_{13} \overset{P_{27}}{\overline{\wedge}} l_{86} \overset{P_{20}}{\overline{\wedge}} l_2$

A pair of projectivities between two lines can, o course, be combined to give a single projectivity between either line and itself, and if this is done, pairs of corresponding points common to the original projectivities give rise to self-corresponding points in the composition of these projectivities. This suggests the possibility of regarding a projectivity between cobasal ranges in $PG_{10(nd)}$ as an instance of the matching process in the so-called *problème de rencontres*. This problem is stated, typically, in the following facetious way: If a careless secretary types N letters with the corresponding envelopes and then puts the letters in the envelopes in a completely random way, what is the probability that exactly n of the letters are placed in the right envelopes? The problem of computing these probabilities is not difficult and its solution can be found in [3]. The main point of this note is to call attention to the surprising fact that according to our data, self-corresponding points in randomly generated projectivities between cobasal ranges in $PG_{10(nd)}$ occur with probabilities which are very close to those predicted by the *problème de rencontres*, as shown in Table 2.

TABLE 2

Total number of projectivities	$n=0$	1	2	3	4	5	6	7
	Number of projectivities with n self-corresponding points							
1521	574*	530	287	94	29	6	1	0
	.3774	.3485	.1887	.0618	.0191	.0039	.0007	.0000
2926	1065	1068	542	188	51	9	3	0
	.3640	.3650	.1852	.0643	.0174	.0031	.0010	.0000
4500	1628	1668	822	290	74	13	4	1
	.3618	.3707	.1827	.0644	.0164	.0029	.0009	.0002
6000	2178	2209	1112	370	104	21	5	1
	.3630	.3682	.1853	.0617	.0173	.0035	.0008	.0002
Frequencies for random matching	.3679	.3679	.1834	.0613	.0153	.0031	.0005	.0001

* Each cell contains first the number of projectivities of the indicated type and then their relative frequency.

The significance of this observation escapes us at present.

References

1. Steven Heath, Finite Projective Geometries, Master's Thesis, University of Utah, Salt Lake City, 1967.
2. C. R. Wylie, Introduction to Projective Geometry, McGraw-Hill, New York, 1970, pp. 514–575.
3. William Feller, Probability Theory and its Application, vol. I, Wiley, New York, 1950, p. 66.

A LINEAR FORM RESULT IN THE GEOMETRY OF NUMBERS

L. J. MORDELL, University of Calgary, Canada

A result included in Minkowski's classical theorem [1] states that a three dimensional closed region symmetrical around the origin and of volume $\geqq 8$ contains a point other than the origin of every lattice of determinant one.

The usual applications are to parallelepipeds and to the region $|x|^r+|y|^r+|z|^r \leqq m$, $r \geqq 1$. It may be of interest to give another application, namely to a part of a parallelepiped cut off by various planes.

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The usual applications are to parallelepipeds and to the region $|x|^r+|y|^r+|z|^r \leqq m$, $r \geqq 1$. It may be of interest to give another application, namely to a part of a parallelepiped cut off by various planes.

THEOREM. *The region* R,

$$|x| \leqq a,\ |y| \leqq a,\ |z| \leqq b,\ |x-y| + |z| \leqq c,$$

contains a point not the origin of every lattice of determinant one if $2a > c$ *and either* $b \geqq c \geqq \sqrt[3]{6}$ *or* $c \geqq b \geqq \sqrt[3]{6}$.

The condition $2a > c$ is imposed to exclude the possibility of a sublattice in x, y of determinant one, for if $2a \leqq c$, there may be a trivial solution with $z = 0$ if $a \geqq 1$.

We have to express the condition that R has volume $V \geqq 8$. Make the substitution $x\sqrt{2} \to x + y$, $y\sqrt{2} \to x - y$. Then R becomes the region,

$$|x+y| \leqq a\sqrt{2},\ |x-y| \leqq a\sqrt{2},\ |z| \leqq b,\ |y\sqrt{2}| + |z| \leqq c.$$

The first two inequalities are equivalent to $|x| + |y| \leqq a\sqrt{2}$. Hence $V = 8V_1$, where V_1 is the volume of the region,

$$x \geqq 0,\ y \geqq 0,\ z \geqq 0,\ x + y \leqq a\sqrt{2},\ z \leqq b,\ y\sqrt{2} + z \leqq c.$$

We first require the area of the x, y region,

$$x \geqq 0,\ y \geqq 0,\ x + y \leqq a\sqrt{2},\ y\sqrt{2} \leqq c - z.$$

We cannot have $(c-z)/\sqrt{2} \geqq a\sqrt{2}$, since $c < 2a$. Hence the x, y region is a trapezium of height $(c-z)/\sqrt{2}$ and so of area

$$\frac{(c-z)}{2\sqrt{2}}\left(a\sqrt{2} + a\sqrt{2} - \frac{c-z}{\sqrt{2}}\right) = \frac{1}{2}(c-z)\left(2a - \frac{c-z}{2}\right).$$

We have to integrate this for z from 0 to $\min(b,c)$. Suppose first that $b \geqq c$. Then

$$V_1 = \frac{a}{2}c^2 - \frac{1}{12}c^3 = \frac{6ac^2 - c^3}{12} > 1$$

if $c^3/6 \geqq 1$ since $2a > c$. Suppose next that $c \geqq b$. Then

$$\begin{aligned} V_1 &= \frac{a}{2}(c^2 - (c-b)^2) - \frac{1}{12}(c^3 - (c-b)^3) \\ &= \frac{a}{2}(2bc - b^2) - \frac{1}{12}(3c^2 b - 3cb^2 + b^3) \\ &> \frac{bc^2}{2} - \frac{b^2 c}{4} - \frac{bc^2}{4} + \frac{b^2 c}{4} - \frac{b^3}{12} = \frac{bc^2}{4} - \frac{b^3}{12} \\ &> 1 \text{ if } b^3/6 \geqq 1 \text{ since } c \geqq b. \end{aligned}$$

A more general region is

$$|x_r| \leqq a, (r = 1,\cdots,2n),\ |x_{2n+1}| \leqq b,\ \sum_{r=1}^{n} |x_{2r-1} - x_{2r}| + |x_{2n+1}| \leqq c,$$

and can be dealt with similarly.

In a paper to appear in the Sierpinski volume of Acta Arithmetica, I have dealt with the region derived from the above by the omission of the x_{2n+1} term.

Reference

1. G. H. Hardy and E. M. Wright, The Theory of Numbers, Oxford University Press, New York, 1962, p. 394.

AN "OBVIOUS" BUT USEFUL THEOREM ABOUT CLOSED CURVES

JONATHAN SCHAER, University of Calgary, Alberta, Canada

Let c be a closed, continuous and rectifiable curve in the Euclidean plane E, and $\partial\tilde{c}$ the boundary of its convex hull $\tilde{c}$; $\partial\tilde{c}$ is also a closed, continuous, and rectifiable curve. Then

$$l(c) \geqq l(\partial\tilde{c})$$

where $l(c)$ denotes the length of c; $l(c) = l(\partial\tilde{c})$ if and only if $c = \partial\tilde{c}$.

Proof. (1) We shall first prove the result, when $\partial\tilde{c}$ is a polygon. This includes the case that c itself is a polygon, for which the result is claimed to be well known [**1**]. Let V denote the set of vertices of $\partial\tilde{c}$, and K the set of all polygons with $|V|$ vertices and V as set of vertices. We shall show that

(a) if $c \notin K$ then there exists a polygon $p \in K$ which is shorter than c, and

(b) if $c \in K$ then we can find a shorter polygon $c_1 \in K$ unless $c = \partial\tilde{c}$. The existence of a shortest polygon in K is evident because K is finite, and we shall have proved that $\partial\tilde{c}$ is (strictly) shorter than any other closed curve containing V.

Let $c = f(I)$, where I is the unit interval $[0,1]$ and $f: I \to E$ is a parametrization of c, $f(0) = f(1)$. Since $V \subset c$ there are $|V|$ values $t_i \in I$ so that $t_1 < t_2 < \cdots < t_{|V|}$, $v_i = f(t_i) \in V$ and $v_i \neq v_j$, for $i \neq j$ $(i,j = 1,2,\cdots,|V|)$.

To prove (a) we simply observe that the polygon $p \in K$ which connects $v_1, v_2, \cdots, v_n, v_1$ in that order is shorter than c.

If $c \in K$, but $c \neq \partial\tilde{c}$, then c passes through V in a different order from $\partial\tilde{c}$. Hence there are i and j so that the edges v_iv_{i+1} and v_jv_{j+1} $(i+1<j)$ of c are intersecting chords of the convex polygon $\partial\tilde{c}$. The polygon c_1 which passes through V in the order

$$v_1, v_2, \cdots, v_{i-1}, v_i, v_j, v_{j-1}, \cdots, v_{i+1}, v_{j+1}, \cdots, v_1$$

is shorter than c since the sum of the lengths of the diagonals is greater than the sum of the lengths of the opposite sides of the quadrilateral $v_iv_jv_{i+1}v_{j+1}$.

(2) Now if c is an arbitrary closed and continuous curve of length 1, say, we can approximate it by polygons p_n, whose vertices are, along c, 2^{-n} apart. From (1) we know that $l(\partial\tilde{p}_n) \leqq l(p_n)$, and it is clear that $\lim_{n\to\infty} l(p_n) = 1$. In order to establish the inequality it is therefore sufficient to prove that $\lim_{n\to\infty} l(\partial\tilde{p}_n) = l(\partial\tilde{c})$. Obviously $\tilde{p}_n \subset \tilde{c}$. And for any point $A \in c$ there exists a point (actually a vertex)

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$B \in p_n$, such that the distance $d(A,B) \leqq 2^{-n-1}$; and if $X \in \tilde{c}$, then X is a convex linear combination of points $A_1, A_2, \cdots \in c$, and if Y is the same linear combination of the corresponding points $B_1, B_2, \cdots \in p_n$, then also $d(X,Y) \leqq 2^{-n-1}$. Thus the Blaschke distance $\delta(\tilde{p}_n, \tilde{c}) \leqq 2^{-n-1}$, and, as $n \to \infty$, $\tilde{p}_n \to \tilde{c}_n$, $\partial\tilde{p}_n \to \partial\tilde{c}$, and finally $l(\partial\tilde{p}_n) \to l(\partial\tilde{c})$.

(3) If $\partial\tilde{c} \neq c$, then $\partial\tilde{p}_n \neq p_n$ for some n, and hence by (1) $l(\partial\tilde{p}_n) < l(p_n)$. We shall show that for $m > n$, $l(p_m) - l(\partial\tilde{p}_m) \geqq l(p_n) - l(\partial\tilde{p}_n)$, and so prove the second part of the theorem. Let p'_k denote a sequence of approximating polygons containing the sequence p_n as a subsequence: $p_k = p'_{2^k}$, and which is obtained by introducing one new vertex at a time. It will be enough to prove the last inequality for the sequence p'_n instead of p_n. Let Q be the new vertex of p'_{k+1}, introduced between the vertices P, R of p'_k. If $Q \in \tilde{p}'_k$ the result is obvious. If however $Q \notin \tilde{p}'_k$, i.e., Q is a vertex of $\partial\tilde{p}'_{k+1}$, denote by S, T the intersections of the segments PQ, QR with $\partial\tilde{p}'_k$ and by U, V the adjacent vertices of $\partial p'_{k+1}$. See the figure.

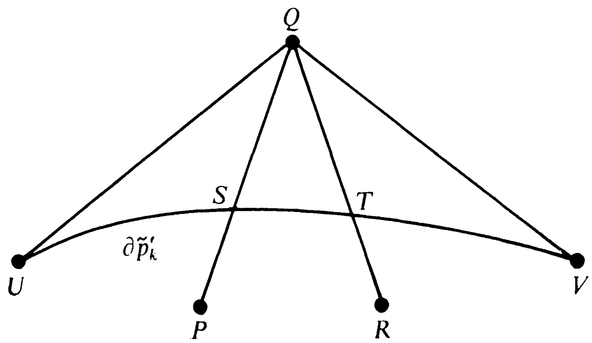

FIG. 1

Now

$$l(\partial\tilde{p}'_{k+1}) - l(\partial\tilde{p}'_k) \leqq d(U,Q) + d(Q,V) - d(U,S) - d(S,T) - d(T,V)$$

and

$$l(p'_{k+1}) - l(p'_k) = d(P,Q) + d(Q,R) - d(P,R).$$

Hence

$$\begin{aligned}[l(p'_{k+1}) - l(\partial\tilde{p}'_{k+1})] - [l(p'_k) - l(\partial\tilde{p}'_k)] &\geqq d(P,S) + d(S,Q) + d(Q,T) + \\ &+ d(T,R) - d(P,R) - d(U,Q) - d(Q,V) + d(U,S) + d(S,T) + d(T,V) \\ &= [d(U,S) + d(S,Q) - d(U,Q)] + [d(Q,T) + d(T,V) - d(Q,V)] + \\ &+ [d(P,S) + d(S,T) + d(T,R) - d(P,R)] \geqq 0.\end{aligned}$$

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A THEOREM ON RATIONAL ZEROS OF A POLYNOMIAL

WALTER LEIGHTON, University of Missouri, Columbia

When a polynomial equation

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \ (a_0 \neq 0)$$

with integral coefficients has an integral root $x = a$, it is well known that $m - a$ must divide $f(m)$ for all integral values of $m \neq a$. So far as the writer knows, the corresponding theorem for rational roots does not appear in the textbooks. It is very useful and may be stated as follows:

THEOREM. *If p/q is a rational root of* (1), *where p and q are relatively prime integers, the number $qm - p$ must divide $f(m)$ for all integral values of m.*

The proof of the theorem is, in a sense, simple. Dividing $f(x)$ by $qx - p$ we have

$$f(x) \equiv (qx - p)u(x), \tag{1}$$

where $u(x)$ is a polynomial with integral coefficients. It follows that

$$f(m) = (qm - p)u(m),$$

for every integer m, and the proof is complete. The observation that the coefficients of $u(x)$ are integers requires attention. This fact is known, but proofs in print do not appear to be readily available. It may be established as follows. We set $y = a_0x$ and obtain the identity

$$a_0^{n-1}f(x) \equiv y^n + a_1y^{n-1} + a_2a_0y^{n-2} + \cdots + a_{n-1}a_0^{n-2}y + a_na_0^{n-1} = g(y), \tag{2}$$

It follows that where $y = a_0x$. Since p/q is a root of (1), $a = a_0p/q$ (an integer) is a zero of $g(y)$.

$$g(y) \equiv \left(y - \frac{a_0p}{q}\right) h(y), \tag{3}$$

where $h(y)$ has integral coefficients with its leading coefficient unity. If we multiply

$$h(y) = y^{n-1} + b_1y^{n-2} + \cdots + b_{n-2}y + b_{n-1}$$

by $y - a$ and set the product equal to $g(y)$ we obtain the equations

$$\begin{aligned} &-ab_{n-1} = a_na_0^{n-1}, \\ &b_{n-1} = ab_{n-2} + a_{n-1}a_0^{n-2},\ b_{n-2} = ab_{n-3} + a_{n-2}a_0^{n-3}, \cdots, b_2 = ab_1 + a_2a_0, \\ &b_1 = a + a_1. \end{aligned} \tag{4}$$

Next, because p/q is a zero of $f(x)$, note that

$$a_0p^n + a_1p^{n-1}q + \cdots + a_{n-1}pq^{n-1} + a_nq^n = 0. \tag{5}$$

It follows that

$$p \mid a_n,\ p^2 \mid (a_{n-1}p + a_nq), \cdots, p^n \mid (a_1p^{n-1} + a_2p^{n-2}q + \cdots + a_{n-1}pq^{n-2} + a_nq^{n-1}). \tag{6}$$

From the first equation in (4) we have that

$$b_{n-1} = \alpha_{n-1}(qa_0^{n-2}),$$

where $\alpha_{n-1} = -a_n/p$ is an integer. From the second equation we have (recalling that $a = a_0p/q$)

$$b_{n-2} = -\left\{\frac{a_{n-1}p + a_nq}{p^2}\right\} qa_0^{n-3};$$

that is, $b_{n-2} = \alpha_{n-2}(qa_0^{n-3})$, where α_{n-2} is an integer. A straightforward induction then establishes that

$$b_{n-j} = -\left(\frac{c_j}{p^j}\right) qa_0^{n-j-1} \qquad (j = 1,2,\cdots n-2), \tag{7}$$

where c_j is the jth right-hand member in (6). The last equation in (4) yields

$$b_1 = a_0\frac{p}{q} + a_1,$$

where, we recall, $q \mid a_0$.

Combining (2) and (3) and replacing y by a_0x we have

$$a_0^{n-1}f(x) \equiv (qx - p)\frac{a_0}{q}h(a_0x).$$

But employing (7) we see that $h(a_0x)$ is readily written as $qa_0^{n-2}h_1(x)$, where $h_1(x)$ is a polynomial with integral coefficients.

The proof of the theorem is complete.

Example. If the equation

$$f(x) \equiv 3x^3 - 7x^2 + 8x - 2 = 0$$

has a rational root p/q, it is one of the numbers

$$\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}, \frac{2}{1}, -\frac{2}{1}, \frac{1}{1}, -\frac{1}{1}. \tag{8}$$

(We may always, of course, assume that $q > 0$.) Setting $m = 1$ we have $f(m) = 2$, and each prospect (8) must satisfy the condition that $(qm - p) \mid f(m)$. Candidates passing this test are

$$\frac{2}{3}, \frac{1}{3}, \frac{2}{1}, -\frac{1}{1}. \tag{9}$$

The test can be repeated for the set (9) with a new choice of m. Indeed, if we employ $m = 2$, there remains only

$$\frac{1}{3},$$

which is, in fact, a root of the equation.

Question. Is there a finite set of values of m such that remaining candidates are necessarily roots of $f(x) = 0$? An admittedly limited set of experiments suggests the possibility that such a set may be all the positive divisors of a_0 and a_n.

This note was written while the author was partially supported by the U. S. Army Research Office (Durham) under Grant DA-ARO-D-31-124-G1007.

COVARIANCE OF MONOTONE FUNCTIONS

JAVAD BEHBOODIAN, University of North Carolina, Chapel Hill

In this note we present and prove an inequality on covariance which seems to be new. This inequality shows that covariance can be interpreted as a measure of the joint variation of two random variables in the same or opposite directions.

THEOREM. *Let f and g be two real functions defined on the range of a random variable X. If they are monotonic in the same direction, then*

$$\operatorname{cov}(f(X), g(X)) \geqq 0; \tag{1}$$

and if they are monotonic in opposite directions, then

$$\operatorname{cov}(f(X), g(X)) \leqq 0. \tag{2}$$

Proof. To prove the inequality (1) we consider two independent and identically distributed random variables X_1 and X_2 with the same distributions as X. Since f and g are monotonic in the same direction,

$$[f(X_1) - f(X_2)][g(X_1) - g(X_2)] \geqq 0 \tag{3}$$

for all points in the sample space. Expanding the left side of (3), taking expectations of both sides of the result, and using the fact that X_1 and X_2 are independent and identically distributed as X, we obtain

$$E[f(X)g(X)] - E[f(X)]E[g(X)] \geqq 0. \tag{4}$$

It is assumed that all the expectations in question exist. Now, by definition of covariance, the inequality (1) follows from the inequality (4) immediately.

We can also prove the inequality (2) by the same argument.

SOME GENERALIZATIONS OF THE PASCAL TRIANGLE

CHARLES CADOGAN, University of Waterloo

1. Introduction. Most students of high school mathematics are familiar with the following relation between the binomial coefficients, namely,

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$$\text{cov}(f(X), g(X)) \geqq 0; \tag{1}$$

and if they are monotonic in opposite directions, then

$$\text{cov}(f(X), g(X)) \leqq 0. \tag{2}$$

Proof. To prove the inequality (1) we consider two independent and identically distributed random variables X_1 and X_2 with the same distributions as X. Since f and g are monotonic in the same direction,

$$[f(X_1) - f(X_2)][g(X_1) - g(X_2)] \geqq 0 \tag{3}$$

for all points in the sample space. Expanding the left side of (3), taking expectations of both sides of the result, and using the fact that X_1 and X_2 are independent and identically distributed as X, we obtain

$$E[f(X)g(X)] - E[f(X)]E[g(X)] \geqq 0. \tag{4}$$

It is assumed that all the expectations in question exist. Now, by definition of covariance, the inequality (1) follows from the inequality (4) immediately.

We can also prove the inequality (2) by the same argument.

SOME GENERALIZATIONS OF THE PASCAL TRIANGLE

CHARLES CADOGAN, University of Waterloo

1. Introduction. Most students of high school mathematics are familiar with the following relation between the binomial coefficients, namely,

Question. Is there a finite set of values of m such that remaining candidates are necessarily roots of $f(x) = 0$? An admittedly limited set of experiments suggests the possibility that such a set may be all the positive divisors of a_0 and a_n.

This note was written while the author was partially supported by the U. S. Army Research Office (Durham) under Grant DA-ARO-D-31-124-G1007.

COVARIANCE OF MONOTONE FUNCTIONS

JAVAD BEHBOODIAN, University of North Carolina, Chapel Hill

In this note we present and prove an inequality on covariance which seems to be new. This inequality shows that covariance can be interpreted as a measure of the joint variation of two random variables in the same or opposite directions.

THEOREM. *Let f and g be two real functions defined on the range of a random variable X. If they are monotonic in the same direction, then*

$$\operatorname{cov}(f(X), g(X)) \geqq 0; \tag{1}$$

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Proof. To prove the inequality (1) we consider two independent and identically distributed random variables X_1 and X_2 with the same distributions as X. Since f and g are monotonic in the same direction,

$$[f(X_1) - f(X_2)][g(X_1) - g(X_2)] \geqq 0 \tag{3}$$

for all points in the sample space. Expanding the left side of (3), taking expectations of both sides of the result, and using the fact that X_1 and X_2 are independent and identically distributed as X, we obtain

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It is assumed that all the expectations in question exist. Now, by definition of covariance, the inequality (1) follows from the inequality (4) immediately.

We can also prove the inequality (2) by the same argument.

SOME GENERALIZATIONS OF THE PASCAL TRIANGLE

CHARLES CADOGAN, University of Waterloo

1. Introduction. Most students of high school mathematics are familiar with the following relation between the binomial coefficients, namely,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \quad (k, n \text{ positive integers}) \tag{1}$$

where

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \quad \text{and} \quad \binom{n}{k} = 0 \quad \text{if} \quad k > n.$$

By using (1) with the initial conditions $\binom{n}{0} = \binom{0}{0} = 1$, the Pascal triangle is easily generated.

Suppose in (1) we replace each term $\binom{s}{r}$ by $f(r,s)$; then we obtain the recurrence relation:

$$f(k,n) = f(k,n-1) + f(k-1,n-1). \tag{2}$$

Let us denote the set of integers, the set of nonnegative integers and the set of real numbers by Z, Z^* and R respectively. The binomial coefficient $\binom{n}{k}$ may now be regarded as a function f defined on the lattice $Z^* \times Z^*$ which satisfies (2) and is uniquely determined by initial values on $Z^* \times \{0\}$ and $\{0\} \times Z^*$ and the Pascal triangle is the set of values of the function f. Viewed in this way, the binomial coefficients are a subclass of a more general class of functions f defined by

$$f: Z \times Z^* \to R \tag{3}$$

satisfying the functional relationship

$$f(k,n) = pf(k,n-1) + qf(k-1,n-1), \quad p,q \in R \tag{4}$$

subject to certain initial conditions $f(k,0) = d_k \in R$.

We now present a theorem on these functions defined by (3) and we apply it in the two special cases in which the initial conditions d_k give rise to (i) an Arithmetic Progression (A.P.) and (ii) a Geometric Progression (G.P.) for $f(0,n)$. We shall call the function values $f(k,n)$, $k \in Z^*$, obtained in these two cases the A.P. "triangle" and G.P. "triangle" respectively.

2. The theorem and applications.

THEOREM 1. $f(k,n) = \sum_{r=0}^{n} \binom{n}{r} p^{n-r} q^r f(k-r,0)$.

Proof. (By induction). The theorem is easily verified for $n = 1$ for all k by using (4). Assume true for all $s \leqq n$, $s \in Z^* - \{0\}$ and for all $k \in Z$.

$$\begin{aligned} f(k,n+1) &= pf(k,n) + qf(k-1,n) \\ &= \sum_{r=0}^{n} \binom{n}{r} p^{n+1-r} q^r f(k-r,0) \\ &\quad + \sum_{r=0}^{n} \binom{n}{r} p^{n-r} q^{r+1} f(k-r-1,0), \end{aligned}$$

$$= p^{n+1}f(k,0) + \sum_{r=1}^{n}\left[\binom{n}{r} + \binom{n}{r-1}\right]p^{n+1-r}q^r f(k-r,0)$$

$$+ q^{n+1}f(k-n-1,0),$$

$$= p^{n+1}f(k,0) + \sum_{r=1}^{n}\binom{n+1}{r}p^{n+1-r}q^r f(k-r,0)$$

$$+ q^{n+1}f(k-n-1,0), \quad \text{by (1)}$$

$$= \sum_{r=0}^{n+1}\binom{n+1}{r}p^{n+1-r}q^r f(k-r,0).$$

Hence true for $\overline{n+1}$ whenever true for $n \in Z^*$.

This completes the proof of the theorem.

COROLLARY. $f(0,n) = \sum_{r=0}^{n} \binom{n}{r} p^{n-r}q^r f(-r,0)$.

To obtain the Pascal triangle and the A.P. and G.P. "triangles", we must put $p = q = 1$ in Theorem 1. Then (2) and (4) are identical, except that in (2) $k \in Z^*$ and in (4) $k \in Z$.

Values for k, n are taken at lattice points on the x-axis and on the nonnegative y-axis respectively in the Cartesian plane and some of the values of $f(k,n)$ are assigned to these points as in Figures 1–3.

APPLICATIONS.

1. The Pascal triangle	2. The A.P. triangle	3. The G.P. triangle
$d_k = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases}$	$\begin{cases} a, & \text{if } k = 0 \\ d, & \text{if } k = -1 \\ 0, & \text{otherwise} \end{cases}$	$\begin{cases} a(r-1)^{-k}, & \text{if } k \leqq 0 \\ 0, & \text{otherwise} \end{cases}$
$f(0,n) = \binom{n}{0} = 1$	$a\binom{n}{0} + d\binom{n}{1}$ $= a + nd$	$a\sum_{r=0}^{n}\binom{n}{r}(r-1)^r$ $= a(1+r-1)^n$ $= ar^n$
$f(k,n) = \binom{n}{k}d_0 = \binom{n}{k}$	$\binom{n}{k}f(0,0)$ $+ \binom{n}{k+1}f(-1,0)$ $= a\binom{n}{k} + d\binom{n}{k+1}$	$a\sum_{r=0}^{n}\binom{n}{r}(r-1)^{r-k}$ $= a\sum_{s=0}^{n-k}\binom{n}{s+k}(s+k-1)_s$

Pascal triangle

			n					
			.	.	.	.	.	.
			1	4	6	4	1	— —
			1	3	3	1	0	— —
			1	2	1	0	0	— —
			1	1	0	0	0	— —
0	0	0	1	0	0	0	0	k

FIG. 1.

The A.P. "triangle"

		.	.	.	.	.	.	
		d	$a + 4d$	$4a + 6d$	$6a + 4d$	$4a + d$	a	— —
		d	$a + 3d$	$3a + 3d$	$3a + d$	a	0	— —
		d	$a + 2d$	$2a + d$	a	0	0	— —
		d	$a + d$	a	0	0	0	— —
0	0	d	a	0	0	0	0	

FIG. 2.

The G.P. "triangle"

.	.	.	.	.	.	.	.
·	$ar^4(r-1)^2$	$ar^4(r-1)$	ar^4	$a(r^2 + r^3 + r + 1)$	$a(r^2 + 2r + 3)$	$a(r + 3)$	a
·	$ar^3(r-1)^2$	$ar^3(r-1)$	ar^3	$a(r^2 + r + 1)$	$a(r + 2)$	a	0
·	$ar^2(r-1)^2$	$ar^2(r-1)$	ar^2	$a(r + 1)$	a	0	0
·	$ar(r-1)^2$	$ar(r-1)$	ar	a	0	0	0
	$a(r-1)^2$	$a(r-1)$	a	0	0	0	0

FIG. 3.

These figures can all be extended by using (2).

With the initial conditions as specified in the applications 1–3 above, respectively, and for arbitrary p, q, $f(k,n)$ is given by

$$f(k,n) = \begin{cases} \binom{n}{k} p^{n-k} q^k, \\ a\binom{n}{k} p^{n-k} q^k + d\binom{n}{k+1} p^{n-k-1} q^{k+1}, \\ a\sum_{s=0}^{n} \binom{n}{k+1} (s+k-1)^s p^{n-k-s} q^{k+s}. \end{cases}$$

3. A generalization. The function f defined in (3) may be generalized to three dimensions in the following way. Let

$$f: Z \times Z \times Z^* \to R$$

where

$$f(k,m,n) = pf(k,m,n-1) + qf(k,m-1,n-1) + rf(k-1,m-1,n-1).$$

Then $f(k,m,n)$ satisfies the following theorem which is stated without proof.

THEOREM 2. *$f(k,m,n) = \sum \binom{n}{u,\ v,\ w} p^u q^v r^w f(k-w, m-v-w, 0)$ where $u+v+w=n$ and*

$$\binom{n}{u,\ v,\ w} = \frac{n!}{u!\,v!\,w!}.$$

With given initial conditions, explicit formulas can be obtained from Theorem 2.

The author wishes to thank the referee for comments which were very useful in improving this paper.

INFINITE COMPLEMENTING SETS

ANDRZEJ MAKOWSKI, Warsaw, Poland

A. M. Vaidya [1] asked whether there exist two infinite sets A, B of nonnegative integers such that every positive integer n has a unique representation in the form $n = a + b$, $a \in A$, $b \in B$. The following example answers this question in the affirmative:

With the initial conditions as specified in the applications 1–3 above, respectively, and for arbitrary p, q, $f(k,n)$ is given by

$$f(k,n) = \begin{cases} \binom{n}{k} p^{n-k} q^{k}, \\ a\binom{n}{k} p^{n-k} q^{k} + d\binom{n}{k+1} p^{n-k-1} q^{k+1}, \\ a\sum_{s=0}^{n} \binom{n}{k+1} (s+k-1)^{s} p^{n-k-s} q^{k+s}. \end{cases}$$

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Let A be the set consisting of 0 and all positive integers of the form $2^{2n_1} + 2^{2n_2} + \cdots + 2^{2n_k}$ (n_i — different nonnegative integers) and B — the set of numbers of the form $2a$, $a \in A$. The required property follows from the uniqueness of binary expansion of an integer.

Reference

1. A. M. Vaidya, On complementing sets of nonnegative integers, this MAGAZINE, 39 (1966) 43–44.

BOOK REVIEWS

EDITED BY D. ELIZABETH KENNEDY, University of Victoria

Materials intended for review should be sent to: Professor D. Elizabeth Kennedy, Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada.

Reviews of texts at the freshman-sophomore level based upon classroom experience will be welcomed by the Book Review Editor.

A boldface capital **C** *in the margin indicates a classroom review.*

Introduction to Ordinary Differential Equations. By Shepley L. Ross. Blaisdell, Waltham-Toronto-London, 1966. viii + 337 pp.

The subject of differential equations constitutes a very useful branch of mathematics. This subject has been, and it continues to be, an area of great theoretical research and practical applications. Some of the most widely used books on ordinnary differential equations generally fall into one of the following three categories: (i) those that tend to concentrate on the theory and ignore the applications, (ii) those that give emphasis to the applications and slight the theory, and (iii) those that are concerned mainly with techniques of solution, so much so that both theory and application receive minimal attention. The present book, however, attempts to avoid fitting into any of the aforementioned categories by providing a more or less balanced presentation of theory, applications, and techniques. It does presuppose a knowledge of elementary calculus. And it encourages the interested reader by including a list of references at the end of each chapter.

This book is essentially the same as the first nine chapters of the author's longer book entitled *Differential Equations* (Blaisdell, 1964), and it is aimed at providing the traditional one-semester introductory course in the subject. Nevertheless, a few less traditional and more specialized topics are also dealt with in significant detail. Considerable emphasis is given, for instance, to approximate methods (including elementary numerical methods) and on the theory and elementary applications of the Laplace transform in the solution of linear differential equations with constant coefficients.

Let A be the set consisting of 0 and all positive integers of the form $2^{2n_1}+2^{2n_2}+\cdots+2^{2n_k}$ (n_i — different nonnegative integers) and B — the set of numbers of the form $2a$, $a \in A$. The required property follows from the uniqueness of binary expansion of an integer.

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Throughout this book the presentation is both clear and logical; significant results are stated precisely as theorems, and various methods of solution are described with considerable explanation. Theorems, definitions, and examples are clearly labeled, and detailed and careful proofs are included for some of the basic theorems.

The first chapter provides an introduction to the subject of differential equations. Its object is to introduce the reader to the fundamental aspects of the subject and at the same time give a brief survey of the origin and applications of differential equations. A discussion of the existence, uniqueness, and nature of solutions is also included. Chapter 2 deals with first-order differential equations for which exact solutions are obtainable by one or the other of the standard methods. Techniques are fully illustrated by means of carefully chosen examples, and numerous exercises are included. An interesting feature here is that the author also includes an exercise involving relevant cases of the Riccati equation.

Chapter 3 is on applications of first-order equations. It deals with problems involving orthogonal and oblique trajectories, problems in mechanics, and rate problems. The author could well have included, in this chapter, trajectory problems involving polar coordinates, and some simple electric-circuit problems.

In Chapter 4 the author gives the basic theory of linear differential equations, and he discusses explicit methods of solving linear equations of order $\geqq 2$. Applications of these equations are considered in Chapter 5, wherein the author does include, among others, several electric-circuit problems.

Series solutions of linear differential equations form the subject-matter of Chapter 6, and Chapter 7 provides an introduction to the theory and solutions of systems of linear differential equations. Some more applications to mechanics and to electric circuits are given here.

Chapter 8 on approximate methods of solving first-order equations and Chapter 9 on the Laplace transform are devoted, as we have already indicated, to these slightly less traditional and more specialized topics in the theory of differential equations. The chapter on the Laplace transform, which happens to be the last chapter of this book, is very well written, and it considers several detailed examples to illustrate how the transform theory can be employed to solve certain initial-value problems involving linear differential equations with constant coefficients.

As usual, answers to odd-numbered exercises in the text are appended. The bibliography that follows contains a list of as many as nineteen books.

It may be of interest to conclude with the remark that this book will prove to be useful if it is chosen for the right clientele. It could well be used as a one-semester textbook if, for instance, the students come from different disciplines of science and engineering.

H. M. SRIVASTAVA, University of Victoria,
Victoria, Canada.

ADVISING MATHEMATICS MAJORS

Faculty advisors of undergraduate majors in mathematics must be especially careful and honest these days in what they say to prospective graduate students about the employment outlook for a young person who goes on for the Ph. D.

Academic employment. The United States has gone very quickly from a scarcity of college teachers to a surplus in almost all academic disciplines, and there is no visible factor that could reverse this prospect in the next 18 years. The children who will go to college in that period have already been born and counted. Beyond 18 years we do not yet have the facts but there is no indication that the declining birth rate will soon reverse the trend and bring new growth to college enrollments. College budgets are already being made in the knowledge of these lowered projections so that many faculty vacancies are being cancelled.

Since over 80 percent of mathematics Ph. D.'s have in the past found employment in college or university teaching, a young person entering graduate work should do so with the severe current and future restriction of academic job apportunities in mind. Competition will necessarily be severe.

Other employment. Employment of Ph. D. mathematicians in the computer industry, other industry, and government laboratories was growing until a few years ago. Now these sectors are depressed. Unlike college teaching, the recession in the computer industry may be temporary, but we cannot now predict when or how much recovery will occur.

An increasing fraction of mathematics students will study mathematics for the reason that they like to do mathematics rather than to qualify for a job. Their Ph. D. degree will give them a certain status and identity, as well as proof of superior ability and achievement. But many of these must seek employment for which mathematics is not a specific qualification. In all walks of life the skills and training of a mathematician should be an advantage to him and to his employer. No other scientific field has such universal applicability in human affairs. But those who go out on their own in this uncertain future must be prepared to assume greater risks than did their earlier counterparts who could confidently look forward to college teaching positions.

Committee on Employment and Educational
Policy of the American Mathematical
Society

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, J. S. FRAME, Michigan State University

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before November 15, 1972

PROPOSALS

831. *Proposed by Gary R. Gruber, Hofstra University, New York.*

Consider the functional equation holding for all x, y such that $F(x,y)$ is bounded and $f(x+y)-f(x)=y^2F(x,y)$. Find an explicit solution (that is, find the functions $f(x)$ and $F(x,y)$, excepting $F(x,0)$ which is indeterminate) and show that the solution is unique.

832. *Proposed by Guy A. R. Guillot, Montreal, Canada.*

A certain six-digit integer when divided by an even two-digit integer gives a result in which the sum of the 1st, 2nd, 3rd and 4th product is equal to the first, second, third and fourth difference digits in the quotient, respectively. If there is no remainder, what is the prime quotient?

```
X X ) X X X X X X ( X X X
      X X X
      -----
        X X X
        X X X
        -----
          X X X
          X X X
          -----
            X X X
            X X X
            -----
```

833. *Proposed by A. K. Gupta, University of Arizona.*

Let $a_n = (n!)^{1/n}$ where n is a positive integer. Consider the sequence $\{b_n\}$ where $b_n = a_{n+1}/a_n$. Show that $b_n > 2^{1/n+1}$ for all $n > 1$.

834. *Proposed by Marion B. Smith, University of Wisconsin, Baraboo, Wisconsin.*

Let c be a positive integer and define a set S_c as follows: $S_c = \{(x,y) \mid x$ and y are positive or negative integers and $1/x + 1/y = 1/c\}$. Prove that $\sum_{(x,y)\in S_c}(x+y) = 4c\tau(c^2)$ where $\tau(c^2)$ is the number of divisors of c^2.

835. *Proposed by Sidney H. L. Kung, University of Jacksonville, Florida.*

Show that:
(a) $\sin(\cos x) < \cos(\sin x)$
(b) $\cos(\sin^{-1}x) < \sin^{-1}(\cos x)$, $0 \leqq x \leqq 1$
(c) $|\sin px| < p\,|\sin x|$ for any integral $p > 1$, $\sin x \neq 0$.

836. *Proposed by R. A. Struble, North Carolina State University at Raleigh.*

Find simple necessary and sufficient conditions on the function $f: R \to C$ from the real line to the complex plane in order that

$$d(a,b) = \sup\{|f(t+a) - f(t+b)| : t \in R\}$$

define a metric which induces the Euclidean topology for R.

837. *Proposed by Vladimir F. Ivanoff, San Carlos, California.*

Prove that the altitudes of any triangle bisect the angles of another triangle whose vertices are the feet of the altitudes of the first triangle.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q542. If a, b, c, d and x, y denote respective lengths of four consecutive sides and both diagonals of a quadrilateral having both an incircle and a circumcircle, show that $(a+b+c+d)^2 \geqq 8xy$, with equality if and only if the quadrilateral is a square.

[*Submitted by Murray S. Klamkin*]

Q543. Show that for all natural numbers $n \geqq 4$, $(n-1)^n > n^{n-1}$.

[*Submitted by Alexander Zujus*]

Q544. Show that 5, 7, 11, 13 is the only prime quadruple in which the arithmetic mean of the members is a perfect square.

[*Submitted by John Hudson Tiner*]

Q545. If x is an integer, prove that the only odd prime divisors of $x^4 + 2x^3 + 3x^2 + 2x - 4$ are of the form $10m \pm 1$.

[*Submitted by Erwin Just*]

Q546. If n is an integer greater than 2, prove that n is the sum of the nth powers of the roots of $x^n - kx - 1 = 0$.

[*Submitted by Erwin Just*]

(Answers on page 176)

SOLUTIONS

Minimum of an Exponential Function

803. [September, 1971] *Proposed by Kenneth Rosen, University of Michigan.*

Let x and y be positive real numbers with $x + y = 1$. Prove that $x^x + y^y \geqq \sqrt{2}$ and discuss conditions for equality.

I. *Solution by Leon Bankoff, Los Angeles, California.*

We may write

$$f(x) = x^x + (1-x)^{1-x}, \text{ where } 0 < x < 1.$$

$$f'(x) = x^x(1 + \log x) - (1-x)^{1-x}(1 + \log[1-x]).$$

$$f''(x) = x^{x-1} + x^x(1 + \log x)^2 + (1-x)^{-x} + (1 + \log[1-x])^2.$$

Since $f''(x)$ is positive for $0 < x < 1$, $f(x)$ is *minimum* when $f'(x) = 0$, that is, when $x = 1 - x = \frac{1}{2}$. As a result, $x^x + (1-x)^{1-x} \geqq \sqrt{\frac{1}{2}} + \sqrt{\frac{1}{2}} = \sqrt{2}$.

II. *Solution by Leon Bankoff, Los Angeles, California.*

By the Arithmetic-Geometric mean inequality, $x^x + y^y \geqq 2\sqrt{x^x y^y}$, equality holding when $x^x = y^y$, that is, when $x = y = \frac{1}{2}$. Hence $x^x + y^y \geqq 2\sqrt{\frac{1}{2}} = \sqrt{2}$.

III. *Solution by Murray S. Klamkin, Ford Motor Company.*

It is well known that if $F(x)$ is strictly convex for $0 \leqq x \leqq a$, then

$$F(x_1) + F(x_2) + \cdots + F(x_n) \geqq nF\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)$$

with equality iff $x_1 = x_2 = \cdots = x_n$. Since $D^2 x^x = x^x(1+\log x)^2 + x^{x-1}$, x^x is strictly convex for $x \geqq 0$. Thus for $x_1 + x_2 + \cdots + x_n = nb$,

$$\sum_{i=1}^{n} x_i^{x_i} \geqq nb^b.$$

The given problem corresponds to the special case $n = 2$, $b = \frac{1}{2}$.

Also solved by Merrill Barnebey, University of Wisconsin at LaCrosse; Stephen D. Brown, Southern Colorado State College; Martine J. Brown, University of Kentucky; Kenneth Campbell, Marshalltown, Iowa; Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, Australia; Robert G. Griswold, University of Hawaii; D. F. Hayes, San Bruno, California; J. A. H. Hunter, Toronto, Canada; Vaclav, Konecny, Jarvis Christian College, Texas; Lew Kowarski, Morgan State College, Maryland; N. J. Kuenzi, Oshkosh, Wisconsin; Shiv Kumar and Miss Nirmal (jointly), Ohio University; Peter A. Lindstrom, Genesee Community College, New York; William Nuesslein, SUNY at Geneseo, New York; Joseph O'Rourke, Saint Joseph's College, Pennsylvania; Albert J. Patsche, U. S. Army Weapons Command, Rock Island, Illinois; C. B. A. Peck, State College, Pennsylvania; Aron Pinker, Frostburg State College, Maryland; William Reese, Olmsted Falls Middle School, Ohio; George A. Roberts, Wiley College, Texas; Louis M. Rotando, Westchester College, New York; Rina Rubenfeld, New York City Community College; E. F. Schmeichel, Itasca, Illinois; Alan Wayne, The Cooper Union, New York; Kenneth M. Wilke, Topeka, Kansas; Robert L. Young, Cape Cod Community College, Massachusetts; and the proposer.

Forty Golf Foursomes

804.* [September, 1971] *Proposed by Zalman Usiskin, University of Chicago.*

Forty golfers play each week ten foursomes. Thus in thirteen weeks it is possible for a particular golfer to play every other golfer. Is it possible for every golfer to play in a foursome with every other golfer in this minimal length of time?

I. *Solution by Michael Goldberg, Washington, D.C.*

Yes. If the players are designated by the numbers 0 to 38, and an asterisk*, then the schedule of the ten foursomes for the first week is shown below:

				(*	0	13	26)				
(1	8	25	9)	(2	16	11	10)	(4	32	22	20)
(14	38	18	21)	(15	24	23	29)	(17	35	33	6)
(27	31	34	12)	(28	36	3	37)	(30	7	19	9)

For the second week, add 3 to each of the numbers and reduce them modulo 39. The asterisk remains unchanged. For the third week, add 6 to each number. In general, add $3(k-1)$ to each number for the schedule for the kth week. This will give 13 schedules before repetition.

Such schedules were derived by E. H. Moore in *Tactical Memoranda*, Amer. J. Math., 18 (1896) 264–303. They are generalizations of Kirkman's 15-schoolgirl problem. A k-adic schoolgirl system of index λ in m letters, designated by SGS $[k, \lambda, m]$ is an arrangement of m letters into blocks of k different letters, and in which every set of λ different letters appears only once in the blocks, and the blocks can be divided into m/k schedules in which every schedule contains all the letters. General methods of finding such SGS $[k,\lambda,m]$ systems are given. In particular, the SGS $[4,2,40]$ described above, is based on the display on page 299 of Moore's paper

II. *Solution by Gary Haggard, University of Maine.*

In Marshall Hall's book, *Combinatorial Theory,* we find Theorem 15.3.6. Let $4t+1=p^n$ where p is a prime and let x be a primitive root of $GF(p^n)$. Then there exists a pair of odd integers c and d such that $(x^c+1)/(x^c-1)=x^d$. Then the blocks

$$(x_1^{2i}, x_1^{2t+2i}, x_2^{2i+c}, x_2^{2t+2i+c}),$$

$$(x_2^{2i}, x_2^{2t+2i}, x_3^{2i+c}, x_3^{2t+2i+c}),$$

$$(x_3^{2i}, x_3^{2t+2i}, x_1^{2i+c}, x_1^{2t+2i+c}),$$

$$(\infty,(0)_1,(0)_2,(0)_3) \quad \text{for } i=0,1,\cdots t-1$$

form a base with respect to A, the additive group of $GF(p^n)$, of a design with $v=12t+4$, $b=(3t+1)(4t+1)$, $r=(4t+1)$, $k=4$, and $\lambda=1$.

For our problem $t=3$ and we can let $c=3$, then for $0\leqq m\leqq 12$ the $(m+1)$-round has the following pairings:

$$(x_1^{2i+m}, x_1^{6+2i+m}, x_2^{2i+3+m}, x_2^{9+2i+m}),$$

$$(x_2^{2i+m}, x_2^{6+2i+m}, x_3^{2i+3+m}, x_3^{9+2i+m}),$$

$$(x_3^{2i+m}, x_3^{6+2i+m}, x_1^{2i+3+m}, x_1^{9+2i}),$$

$$(\infty,(0+m)_1,(0+m)_2,(0+m)_3) \quad \text{for } i=0,1,2.$$

All addition in the exponents is done modulo 13 and we assume the following identifications: $(0+l)_k\equiv x_k^l$ and $x_k^{13}\equiv(0)_k$ for $0\leqq l\leqq 12$ and $1\leqq k\leqq 3$. Also the 40 golfers are divided arbitrarily into four sets

$$\text{I} = \{(0)_1, x_1^1, x_1^2,\cdots x_1^{12}\},$$

$$\text{II} = \{(0)_2, x_2^1, x_2^2,\cdots x_2^{12}\},$$

$$\text{III} = \{(0)_3, x_3^1, x_3^2,\cdots x_3^{12}\},$$

$$\text{IV} = \{\infty\}.$$

By using Theorem 15.3.6 the same type of problem can be solved for other v. An obvious question is what happens when $t=5$, and $v=64$?

The paper, *On balanced incomplete block designs*, by R. C. Bose and S. S. Shrikhande which appeared in the Canadian Journal of Mathematics in 1960 includes a long list of resolvable BIBD's with $k=4$. For $v=64$ there is an RBIBD.

Also solved by John Oman, University of Wisconsin, Oshkosh; and Arthur T. White, Michigan University.

A Unique Triangular Number

805. [September, 1971] *Proposed by Charles W. Trigg, San Diego, California*

Find the unique triangular number Δ_n which is a permutation of the ten digits and for which n has the form $abbb$.

Solution by Nigel F. Nettheim, Toronto, Canada.

Since Δ_n has ten digits, $a \geqq 4$. Now

$$n \cdot (n+1) = 2 \cdot \Delta_n$$

where Δ_n is a permutation of the ten digits. Consider the well-known method of "casting out nines": since the sum of the digits of Δ_n is 45, the remainder after casting out nines on the right hand side is zero. Therefore either n or $(n+1)$ has zero remainder after casting out nines; that is, nine divides $(a+4b)$ or $(a+4b+1)$. Only 12 pairs (a,b) survive this test, namely

$$(8,0), (9,0), (4,1), (5,1), (9,2), (5,3),$$

$$(6,3), (6,5), (7,5), (8,7), (4,8), (8,9).$$

Although further reduction can be effected in this set, it is already convenient to calculate all the corresponding values Δ_n, yielding $n = 75{,}555$ and $\Delta_n = 2{,}854{,}316{,}790$.

Also solved by Merrill Barnebey, University of Wisconsin at LaCrosse; Dermott A. Breault, Microsystems Technology Corporation, Burlington, Massachusetts; Romae J. Cormier, DeKalb, Illinois; Edward A. Cygan, Jr., University of Illinois; J. A. Hunter, Toronto, Canada; Lew Kowarski, Morgan State College, Maryland; Thomas E. Moore, Bridgewater State College, Massachusetts; Nigel F. Nettheim, Toronto, Canada; William Nuesslein, SUNY at Geneseo, New York; C. B. A. Peck, State College, Pennsylvania; William Reese, Olmsted Falls Middle School, Ohio; E. F. Schmeichel, Itasca, Illinois; E. P. Starke, Plainfield, New Jersey; Graham C. Thompson, Binghamton, New York; Zalman Usiskin, University of Chicago; Lowell T. Van Tassel, San Diego City College, California; Kenneth M. Wilke, Topeka, Kansas; and the proposer.

Symmetry About a Line

806. [September, 1971] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let H be the orthocenter of an isosceles triangle ABC, and let AH, BH, and CH intersect the opposite sides in D, E, and F, respectively. Prove that the incenters of the right triangles HBD, HDC, HCE, HEA, HAF, and HFB lie on a conic.

Solution by Vladimir F. Ivanoff, San Carlos, California.

The problem is a special case of the following theorem:

If six points are symmetric about a line, they lie on a conic.

It can be easily proved by the converse of Pascal's theorem, or else by analytical method.

Incidentally, the theorem holds true, if six points are symmetric about a point.

By choosing the point of symmetry as the origin, the six points have coordinates as follows:

$$(x_1, y_1), \quad (x_2, y_2), \quad (x_3, y_3),$$
$$(-x_1, -y_1), (-x_2, -y_2), (-x_3, -y_3,),$$

and the equation of the conic is

$$\begin{vmatrix} x^2 & y^2 & xy & 1 \\ x_1^2 & y_1^2 & x_1 y_1 & 1 \\ x_2^2 & y_2^2 & x_2 y_2 & 1 \\ x_3^2 & y_3^2 & x_3 y_3 & 1 \end{vmatrix} = 0.$$

Also solved by Leon Bankoff, Los Angeles, California (three solutions); Ragnar Dybvik, Tingvoll, Norway; Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, Australia; and the proposer.

Arithmetic-Geometric Mean Inequality

807. [September, 1971] *Proposed by Norman Schaumberger, Bronx Community College.*

Let (x_i), $i = 1,2,3 \cdots$ be an arbitrary sequence of positive real numbers, and set

$$\Delta_k = (1/k) \sum_{i=1}^{k} x_i - \left(\prod_{i=1}^{k} x_i \right)^{1/k}$$

If $n \geqq m$, prove that $n\Delta_n \geqq m\Delta_m$.

I. *Solution by Vaclav Konecny, Jarvis Christian College, Texas, and Technical University of Brno, Czechoslavakia.*

It is sufficient to show that $n\Delta_n - (n-1)\Delta_{n-1} \geqq 0$ for n (integer) > 1.

Consider $P(x_n) = x_n + (n-1)(p_{n-1})^{1/(n-1)} - n(p_n - 1)^{1/n} x_n^{1/n}$, where $P_{n-1} = \prod_{i=1}^{n-1} x_i > 0$ and constant for fixed parameter $n > 1$. $P'(x_n) = 1 - p_{n-1}^{1/n} \ x_n^{1/n-1}$ $P''(x_n) = (1 - 1/n) p_{n-1}{}^{1/n} x_n{}^{1/n-2}$. As $P''(x_n) > 0$ $P(x_n)$ has only one minimum for $x_n > 0$, namely, at $x_n = P_{n-1}^{1/(n-1)}$. But $P(p_{n-1}^{1/(n-1)}) = 0$. Therefore $P(x_n) \geqq 0$, which is the asserted inequality in this solution.

II. *Solution by Murray S. Klamkin, Ford Motor Company.*

It suffices just to prove the case $m = n - 1$ $(n \geqq 2)$, i.e.,

(1) $$(n-1)(x_1 x_2 \cdots x_{n-1})^{1/(n-1)} \geqq n(x_1 x_2 \cdots x_n)^{1/n} - x_n.$$

Let

$$x_n = \lambda^n (x_1 x_2 \cdots x_{n-1})^{1/(n-1)}$$

so that (1) becomes

$$\lambda^n - 1 \geqq n(\lambda - 1)$$

which is a known elementary result (just factor $\lambda^n - 1$) with equality *iff* $\lambda = 1$.

REMARK. Since $\Delta_1 = 0$, the above solution provides an apparently new elementary inductive proof of the Arithmetic-Geometric mean inequality ($\Delta_n \geqq 0$).

Also solved by L. Carlitz, Duke University; Ralph Garfield, The College of Insurance, New York; N. J. Kuenzi and Bob Prielipp University of Wisconsin at Oshkosh (jointly); R. J. Kuttler, Johns Hopkins University; C. B. A. Peck, State College, Pennsylvania; Rina Rubenfeld, New York City Community College; E. F. Schmeichel, Itasca, Illinois; Robert E. Shafer, University of California, Livermore, California; E. P. Starke, Plainfield, New Jersey; J. Ernest Wilkins, Jr., Howard University; and the proposer.

Integral Evaluation

808. [September, 1971] *Proposed by Anthony J. Strecok, Argonne National Laboratory.*

Evaluate the integral

$$I(x) = \int_0^{2\pi} e^{-x/\cos^2\theta}\, d\theta.$$

Solution by L. Carlitz, Duke University.

Put

$$\cos 2\theta = \frac{1 - r^2}{1 + r^2}, \quad \cos^2\theta = \frac{1}{1 + r^2}.$$

For $0 \leqq \theta \leqq \pi/2$ we have $0 \leqq r \leqq \infty$. Also

$$\sin 2\theta\, d\theta = \frac{2r dr}{(1 + r^2)^2}, \quad d\theta = \frac{dr}{1 + r^2}.$$

Then

$$\begin{aligned} I(x) &= \int_0^{2\pi} e^{-x/\cos^2\theta} d\theta \\ &= 4\int_0^{\pi/2} e^{-x/\cos^2\theta} d\theta \\ &= 4\int_0^{\infty} e^{-x(1+r^2)} \frac{dr}{1 + r^2} \end{aligned}$$

so that

$$\begin{aligned} I'(x) &= -4\int_0^{\infty} e^{-x(1+r^2)} dr \\ &= -4e^{-x}\int_0^{\infty} e^{-xr^2} dr \end{aligned}$$

$$= -4x^{-\frac{1}{2}}e^{-x}\int_0^\infty e^{-r^2}dr.$$

Since

$$\int_0^\infty e^{-r^2}dr = \tfrac{1}{2}\sqrt{\pi},$$

we have

$$I'(x) = -2\pi^{\frac{1}{2}}x^{-\frac{1}{2}}e^{-x}.$$

Since $I(\infty) = 0$, it follows that

$$I(x) = 2^{\frac{1}{2}}\int_x^\infty e^{-u}u^{-\frac{1}{2}}\,du$$

$$= 2\pi - 2\pi^{\frac{1}{2}}\int_0^x e^{-u}u^{-\frac{1}{2}}du.$$

Thus $I(x)$ is very closely related to the error function.

Also solved by Donald R. Childs, Naval Underwater Systems Center, Rhode Island; Joseph B. Dance, Florida State University and Dennis J. Diestler, Lafayette, Indiana (jointly); Vaclav Konecny Jarvis Christian College, Texas; Shiv Kumar and Miss Nirmal (jointly), Ohio University; William Nuesslein, SUNY at Geneseo, New York; E. F. Schmeichel, Itasca, Illinois; Robert E. Shafer, University of California, Livermore,California; E. P. Starke, Plainfield, New Jersey; P. D. Thomas, U. S. Naval Oceanographic Office, Suitland, Maryland; J. Ernest Wilkins, Jr., Howard University; K. L. Yocom, South Dakota State University; and the proposer.

Expansion of a Zero Axial Determinant

809. [September, 1971] *Proposed by Furio Alberti, University of Illinois, Chicago Circle.*

It is shown in Muir, *A Treatise on the Theory of Determinants*, that the number of formal terms in the expansion of a zero axial determinant of order n is

$$T = n!\{\tfrac{1}{2}! - \tfrac{1}{3}! + \cdots (-1)^n/n!\}, \qquad n = 2,3,4\cdots$$

Show that $T = [n!/e + \{1 + (-1)^n\}/2]$ where the brackets denote the greatest integer function.

Solution by Joseph O'Rourke, Saint Joseph's College, Pennsylvania.

The infinite series for $1/e$,

$$1/e = \tfrac{1}{2}! - \tfrac{1}{3}! + \cdots (-1)^n/n! + \cdots$$

can be written as

$$1/e = P_n + R_n$$

where

$$P_n = \tfrac{1}{2}! - \tfrac{1}{3}! + \cdots (-1)^n/n!$$

$$R_n = (-1)^{n+1}/(n+1)! + \cdots$$

Hence

$$T = n!\,P_n = n!/e - n!\,R_n.$$

The magnitude of R_n is obviously less than the magnitude of its first term, or $|R_n| < 1/(n+1)!$ Therefore,

$$n!\,|R_n| < 1/(n+1) < 1.$$

Now, for n even, the last term of P_n is positive and $1/e < P_n$; for n odd, $1/e > P_n$. Thus, $n!/e$ is alternately less than and greater than T by an amount smaller than 1. Then $T = [n!/e + 1]$ for n even and $T = [n!/e]$ for n odd. These two equations are equivalent to the single formula

$$T = [n!/e + \{1 + (-1)^n\}/2].$$

Also solved by Richard J. Bonneau, Massachusetts Institute of Technology; Michael Goldberg, Washington, D.C.; M. G. Greening, University of New South Wales, Australia; Vaclav Konecny, Jarvis Christian College, Texas; Shiv Kumar and Miss Nirmal (jointly), Ohio University; William Nuesslein, SUNY at Geneseo, New York; E. F. Schmeichel, Itasca, Illinois; E. P. Starke, Plainfield, New Jersey; Kenneth M. Wilke, Topeka, Kansas; and the proposer.

Comment on Q496

Q496. [January and September, 1971] Find two linearly independent functions whose Wronskian vanishes identically.

[*Submitted by C. Stanley Ogilvy*]

Comment by George Ledin, Jr., University of San Francisco, California.

Ogilvy's example is correct. The Wronskian of Spital's comment should be corrected to read:

$$W = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| + 2x^2\delta(x) \end{vmatrix}.$$

This coincides with Spital's Wronskian everywhere except at $x = 0$. The δ-Dirac function (distribution) $\delta(x)$ occurs as a result of noting that

$|x| = x(2U(x) - 1)$ where $U(x)$ is the "unit-step" function defined

$$U(x) = \begin{cases} 1 & \text{for } x > 0 \\ \tfrac{1}{2} & \text{for } x = 0 \\ 0 & \text{for } x < 0\,. \end{cases}$$

Comment on Q480

Q480 [May, 1970] Prove that $8^n \pm 1$ is composite for every integer $n > 1$.

[*Submitted by E. F. Schmeichel*]

Comment by Charles W. Trigg, San Diego, California.

The proof follows immediately upon the simple factorization $8^n \pm 1 = (2^n)^3 \pm 1 = (2^n \pm 1)(2^{2n} \mp 2^n + 1)$. The restriction $n > 1$ is necessary only for the negative sign.

The same procedure applies also to $(b^k)^n \pm 1$ for k odd and > 1, $b > 2$, $n > 0$. If the double sign be replaced by a minus sign, then k may be even also.

ANSWERS

A542. Since the quadrilateral has an incircle, $a + c = b + d$. Since the quadrilateral is inscribable, $xy = ac + bd$. Thus we must show equivalently that

$$a^2 + c^2 \geqq 2b(a + c - b)$$

For a given $a + c$ the r.h.s. has a maximum value of $(a + c)^2/2$ when $b = (a + c)/2$. Since

$$2(a^2 + c^2) - (a + c)^2 = (a - c)^2 \geqq 0$$

our inequality is established. The stated inequality is also equivalent to

$$(a + b + c + d)^2 \geqq 8(ac + bd)$$

for circumscribable quadrilaterals (a,b,c,d, are consecutive lengths of sides).

A543. If, for $n = k$, $(k - 1)^k > k^{k-1}$ then $(k - 1)^k(k + 1)^k > k^{k-1}(k + 1)^k$ or $(k^2 - 1)^k > k^{k-1}(k + 1)^k$, and, *a fortiori*, $k^{2k} > k^{k-1}(k + 1)^k$.

Therefore $k^{k+1} > (k + 1)^k$. As the statement is true for $n = 4$, $3^4 > 4^3$, the induction is complete and the inequality holds generally.

A544. The arithmetic mean of the members of a prime quadruple is the odd composite number separating the two sets of prime pairs. If N^2 is the odd composite number, then $N^2 - 4$ is the first prime. But $N^2 - 4 = (N - 2)(N + 2)$ which will be composite except for $N^2 = 9$.

A545. If p is an odd prime, then $x^4 + 2x^3 + 3x^2 + 2x - 1 \equiv 0 \pmod{p}$ implies $(x^2 - x + 1)^2 \equiv 5 \pmod{p}$ so that 5 is a quadratic residue of p. Therefore $p = 10m \pm 1$ or $p = 5$. A direct substitution of $x = 0,1,2,3,4 \pmod{5}$, however, shows that $p = 5$ is impossible. The conclusion follows.

A546. Let r_i, $(i = 1,2,3 \cdots n)$ be the roots of $x^n = kx + 1$ and note that since the coefficient of x^{n-1} is zero, $\sum_{i=1}^n r_i = 0$. Then for each i, $r_i^n = kr_i + 1$ which implies $\sum_{i=1}^n r_i^n = \sum_{i=1}^n (kr_i + 1) = k(\sum_{i=1}^n r_i) + n = n$.

(Quickies on page 167)

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